Integrable Particle Dynamics in Accelerators

Tuesday: Accelerators

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Fermilab

Challenges of modern accelerators (the LHC case)

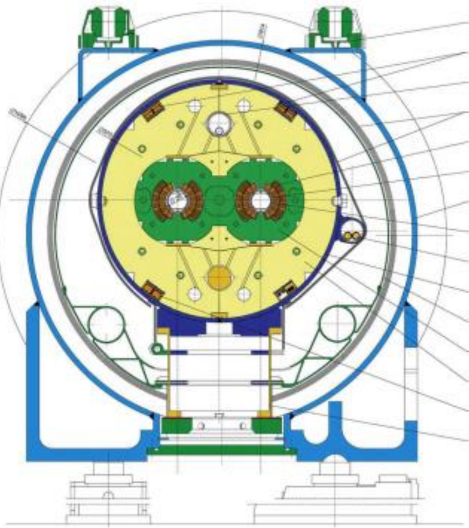
- LHC: 27 km, 7 TeV per beam
  - The total energy stored in the magnets is HUGE: 10 GJ (2,400 kilograms of TNT)
  - The total energy carried by the two beams reaches 700 MJ (173 kilograms of TNT)
  - Loss of only one ten-millionth part (10^{-7}) of the beam is sufficient to quench a superconducting magnet
- LHC vacuum chamber diameter : ~40 mm
- LHC average rms beam size (at 7 TeV): 0.14 mm
- LHC average rms beam angle spread: 2 μrad
  - Very large ratio of forward to transverse momentum
- LHC typical cycle duration: 10 hrs = 4×10^8 revolutions

- Kinetic energy of a typical semi truck at 60 mph: ~7 MJ

What keeps particles stable in an accelerator?

- Particles are confined (focused) by static magnetic fields in vacuum.

- An ideal focusing system in all modern accelerators is nearly integrable
  - There exist 3 conserved quantities (integrals of motion); the integrals are "simple" – polynomial in momentum.
  - The particle motion is confined by these integrals.
  - Nonlinear elements destroy the intregrability at large amplitudes (hence particle losses)



Strong Focusing

The slide reproduces the first page of a journal article:

PHYSICAL REVIEW VOLUME 88, NUMBER 5 DECEMBER 1, 1952

The Strong-Focusing Synchroton—A New High Energy Accelerator*

ERNEST D. COURANT, M. STANLEY LIVINGSTON,† AND HARTLAND S. SNYDER
Brookhaven National Laboratory, Upton, New York

(Received August 21, 1952)

Strong focusing forces result from the alternation of large positive and negative *n*-values in successive sectors of the magnetic guide field in a synchrotron. This sequence of alternately converging and diverging magnetic lenses of equal strength is itself converging, and leads to significant reductions in oscillation amplitude, both for radial and axial displacements. The mechanism of phase-stable synchronous acceleration still applies, with a large reduction in the amplitude of the associated radial synchronous oscillations. To illustrate, a design is proposed for a 30-Bev proton accelerator with an orbit radius of 300 ft, and with a small magnet having an aperture of 1×2 inches. Tolerances on nearly all design parameters are less critical than for the equivalent uniform-*n* machine. A generalization of this focusing principle leads to small, efficient focusing magnets for ion and electron beams. Relations for the focal length of a double-focusing magnet are presented, from which the design parameters for such linear systems can be determined.

BETATRON OSCILLATIONS

RESTORING forces due to radially-decreasing magnetic fields lead to stable "betatron" and "synchrotron" oscillations in synchrotrons. The amplitudes of these oscillations are due to deviations from the equilibrium orbit caused by angular and energy spread in the injected beam, scattering by the residual gas, magnetic inhomogeneities, and frequency errors. The strength of the restoring forces is limited by the

* Work done under the auspices of the AEC.
† Massachusetts Institute of Technology, Cambridge, Massachusetts.

The accelerator Hamiltonian

$$H = c\left[m^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2\right]^{\frac{1}{2}}$$

- After some canonical transformations (see R. Ruth) and in a small-angle approximation

$$H' = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{x^2}{2\rho} + \frac{K}{2}\left(x^2 - y^2\right) - \frac{x\delta}{\rho} + \dots$$

where δ is the relative momentum deviation. For $\delta << 1$:

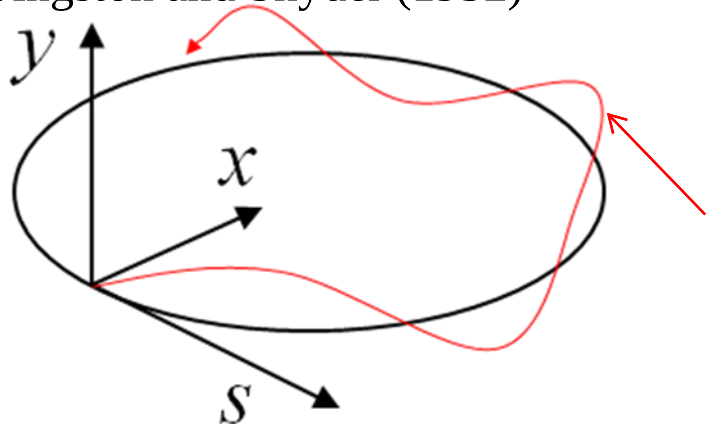
$$H' \approx \frac{p_x^2 + p_y^2}{2} + \frac{K_x(s)x^2}{2} + \frac{K_y(s)y^2}{2}$$

For a pure quadrupole magnet: $K_x(s) = - K_y(s)$

This Hamiltonian is separable!

Strong Focusing - our standard method since 1952

Christofilos (1949); Courant, Livingston and Snyder (1952)

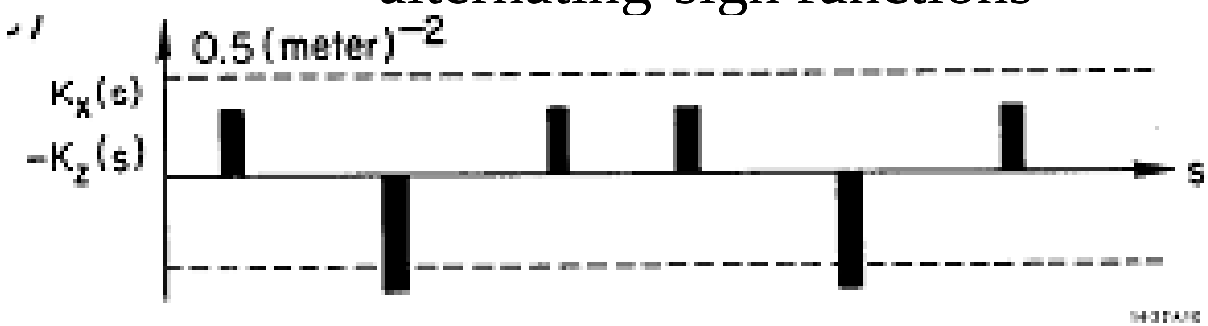


Particle undergoes betatron oscillations

$$\begin{cases} x'' + K_x(s)x = 0 \\ y'' + K_y(s)y = 0 \end{cases}$$

$$K_{x,y}(s + C) = K_{x,y}(s)$$ -- piecewise constant alternating-sign functions

s is "time"



--Magnet lattice and focussing functions in the normal cells of a particular guide field.

Strong focusing

Specifics of accelerator focusing:

- Focusing fields must satisfy Maxwell equations in vacuum

$$\Delta\varphi(x, y, z) = 0$$

- For stationary fields: focusing in one plane while defocusing in another
  - quadrupole: $$\varphi(x, y) \propto x^2 - y^2$$
  - However, alternating quadrupoles results in effective focusing in both planes

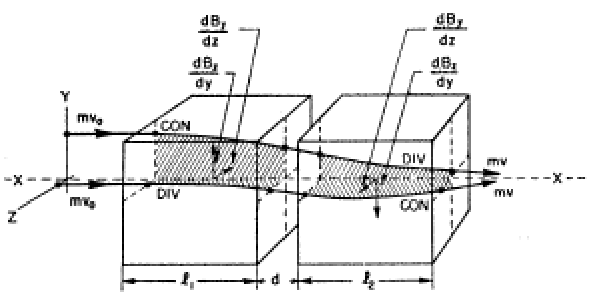


FIG. 8. Illustration of double-focusing in two magnetic lenses with field gradients in opposite directions, showing the alternately convergent and divergent forces and the net convergence of the system.

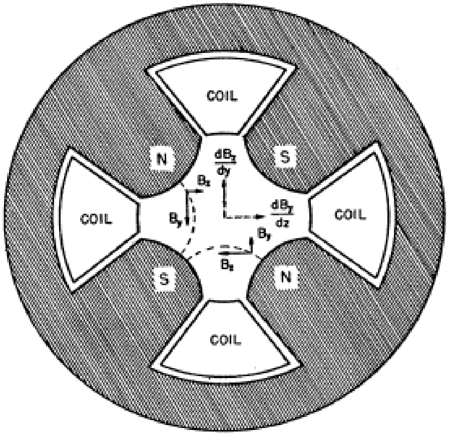
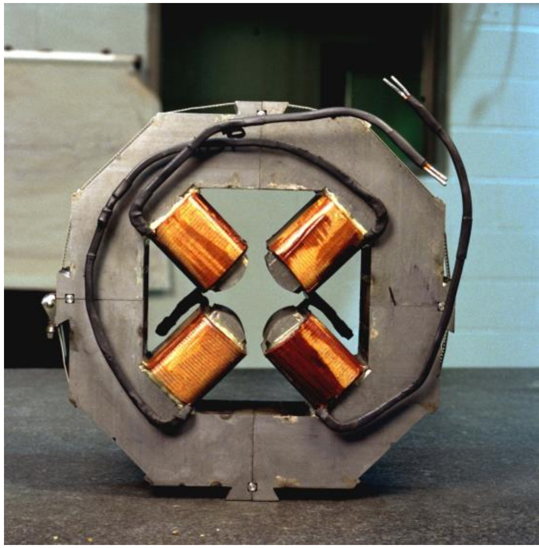


FIG. 9. Cross section of a 4-pole magnet with hyperbolic pole faces to produce uniform and equal field gradients dB_z/dy and dB_y/dz.



Courant-Snyder invariant

Equation of motion for betatron oscillations

$$z'' + K(s)z = 0,$$

$$z = x \text{ or } y$$

$$I_z = \frac{1}{2\beta(s)}\left(z^2 + \left(\frac{\beta'(s)}{2}z - \beta(s)z'\right)^2\right)$$

Invariant (integral) of motion, a conserved qty.

where $\left(\sqrt{\beta}\right)'' + K(s)\sqrt{\beta} = \frac{1}{\sqrt{\beta^3}}$

Action-angle variables

$$H = \frac{p^2}{2} + \frac{K(s)z^2}{2}$$

$$F_1(z,\psi) = -\frac{z^2}{2\beta}\left[\tan\psi - \frac{\beta'}{2}\right]$$

$$I_z = -\frac{\partial F_1}{\partial \psi} = \frac{1}{2\beta(s)}\left(z^2 + \left(\frac{\beta'(s)}{2}z - \beta(s)z'\right)^2\right)$$

$$z = \sqrt{2I_z\beta}\cos\psi$$

See R. Ruth's paper, ref. [2]

$$p = -\sqrt{2I_z/\beta}\left(\sin\psi - \frac{\beta'}{2}\cos\psi\right)$$

$$H_1 = H + \frac{\partial F_1}{\partial s} = \frac{I_z}{\beta(s)}$$

- We can further remove the *s*-dependence by another canonical transformation.

Non-linear elements

$$H_0 = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{K_x(s)x^2}{2} + \frac{K_y(s)y^2}{2}$$

$$H_1 \approx H_0 + \frac{B''(s)}{6B\rho}\left(-3y^2x + x^3\right)$$ sextupole

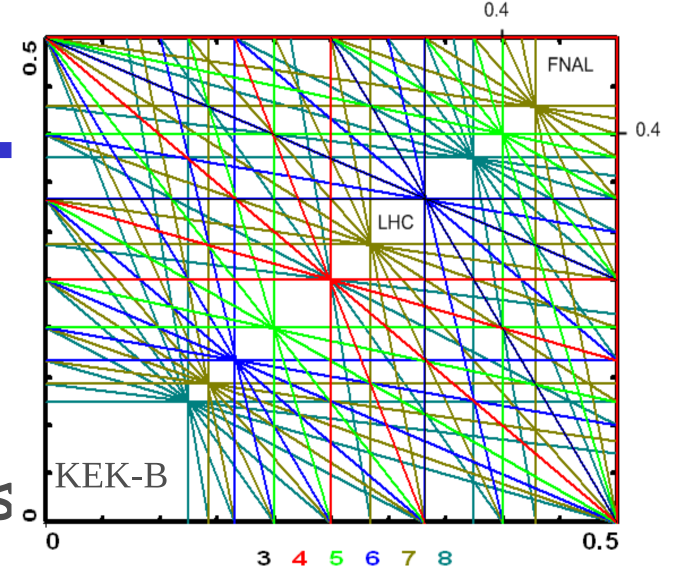
$$H_2 \approx H_0 + \frac{B'''(s)}{24B\rho}\left(x^4 - 6y^2x^2 + y^4\right)$$ octupole

The addition of these nonlinear elements to accelerator focusing (almost always) makes it non-integrable.

Non-linear focusing elements

- It became obvious very early on (~1960), that the use of nonlinear focusing elements in rings is necessary and some nonlinearities are **unavoidable** (magnet aberrations, space-charge forces, beam-beam forces)
  - Sexupoles appeared in 1960s for chromaticity corrections
  - Octupoles were installed in CERN PS in 1959 but not used until 1968. For example, the LHC has ~350 octupoles for Landau damping.
- It was also understood at the same time, that nonlinear focusing elements have both beneficial and detrimental effects, such as:
  - They drive nonlinear resonances (resulting in particle losses) and decrease the dynamic aperture (also particle losses).

KAM theory



- Developed by Kolmogorov, Arnold, Moser (1954-63).
- Explains why we can operate accelerators away from resonances.
- The KAM theory states that if the system is subjected to a weak nonlinear perturbation, some of periodic orbits survive, while others are destroyed. The ones that survive are those that have "sufficiently irrational" frequencies (this is known as the non-resonance condition).
- Does not explain how to get rid of resonances
  - Obviously, for accelerators, making ALL nonlinearities to be ZERO would reduce (or eliminate) resonances
  - However, nonlinearities are necessary and unavoidable.

Nonlinear Integrable Systems

- What we are looking for is a non-linear equivalent of Courant-Snyder invariants, for example something like that,

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{\alpha}{4}\left(x^4 + y^4\right)$$

Specifics of accelerator focusing

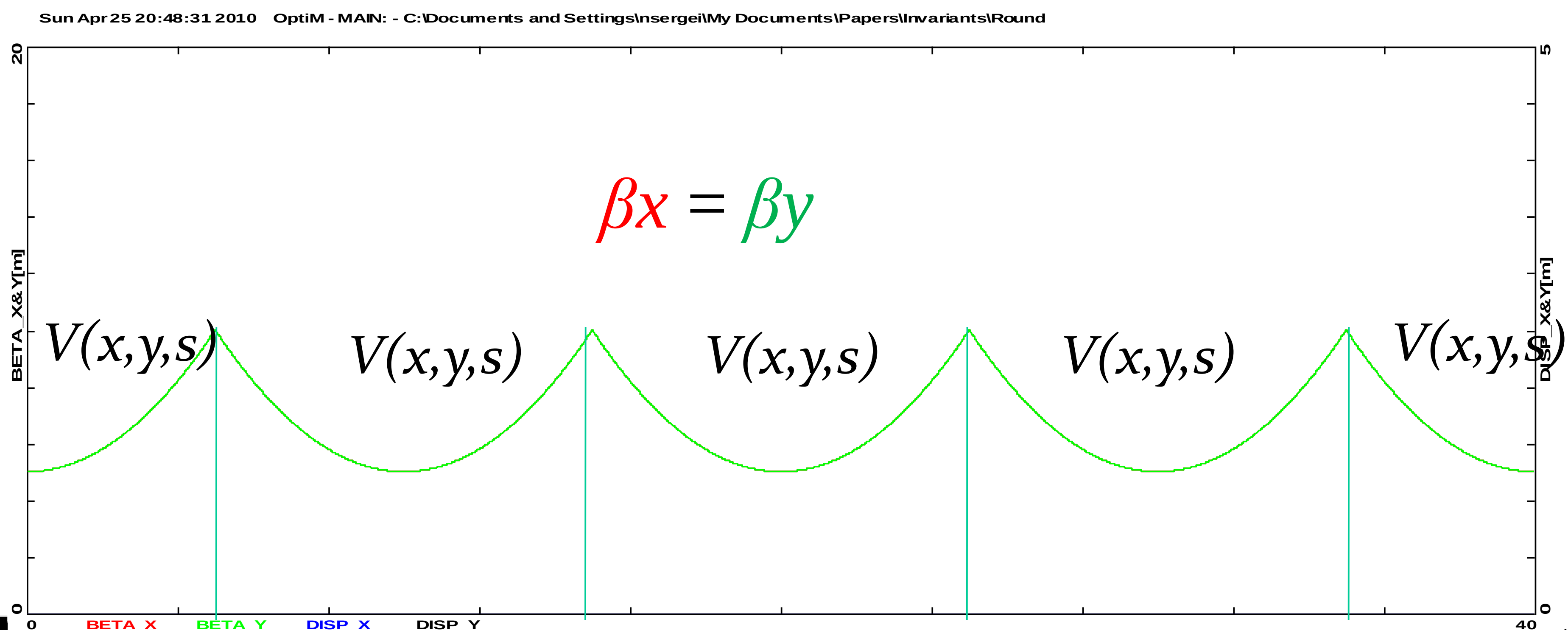
- The transverse focusing system is 2.5D (i.e. time-dependent)
  - In a linear system (strong focusing), the time dependence can be transformed away by introducing a new "time" variable (the betatron phase advance). Thus, we have the Courant-Snyder invariant.
- The focusing elements we use in accelerator must satisfy:
  - The Laplace equation (for static fields in vacuum)
  - The Poisson equation (for devices based on charge distributions, such as electron lenses or beam-beam effects)

First example

See: Phys. Rev. ST Accel. Beams 13, 084002

Start with a round axially-symmetric LINEAR focusing lattice (FOFO)

- Add a special potential *V(x,y,s)* such that it satisfies either the Laplace or the Poisson equation



Special time-dependent potential

Let's consider a Hamiltonian of this FOFO system:

$$H=\frac{p_x^2}{2}+\frac{p_y^2}{2}+K(s)\left(\frac{x^2}{2}+\frac{y^2}{2}\right)+V(x,y,s)$$

where *V(x,y,s)* satisfies the Laplace or the Poisson equations in 2d:

In normalized variables we will have:

$$z_N=\frac{z}{\sqrt{\beta(s)}},$$

$$p_N=p\sqrt{\beta(s)}-\frac{\beta'(s)z}{2\sqrt{\beta(s)}},$$

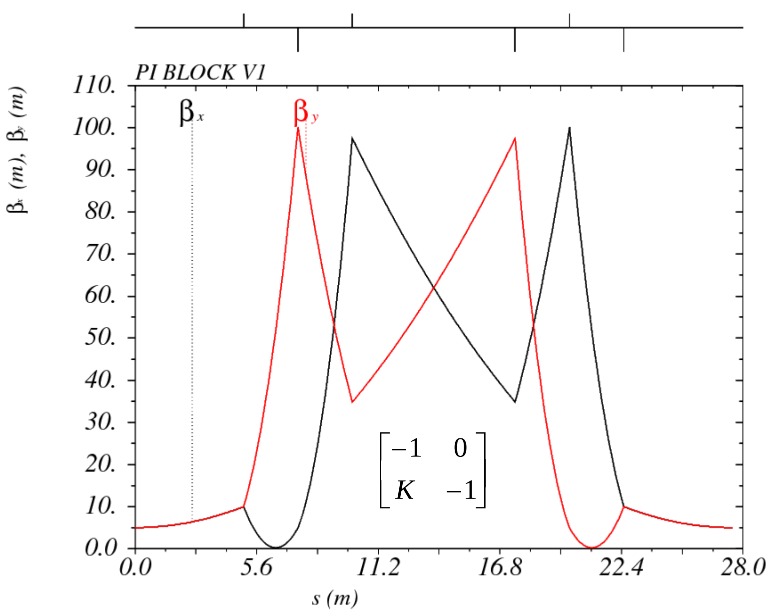
$$H_N=\frac{p_{xN}^2+p_{yN}^2}{2}+\frac{x_N^2+y_N^2}{2}+\beta(\psi)V\left(x_N\sqrt{\beta(\psi)},y_N\sqrt{\beta(\psi)},s(\psi)\right)$$

Where new "time" variable is $\psi(s)=\int_0^s\frac{ds'}{\beta(s')}$

Axially-symmetric focusing lens:

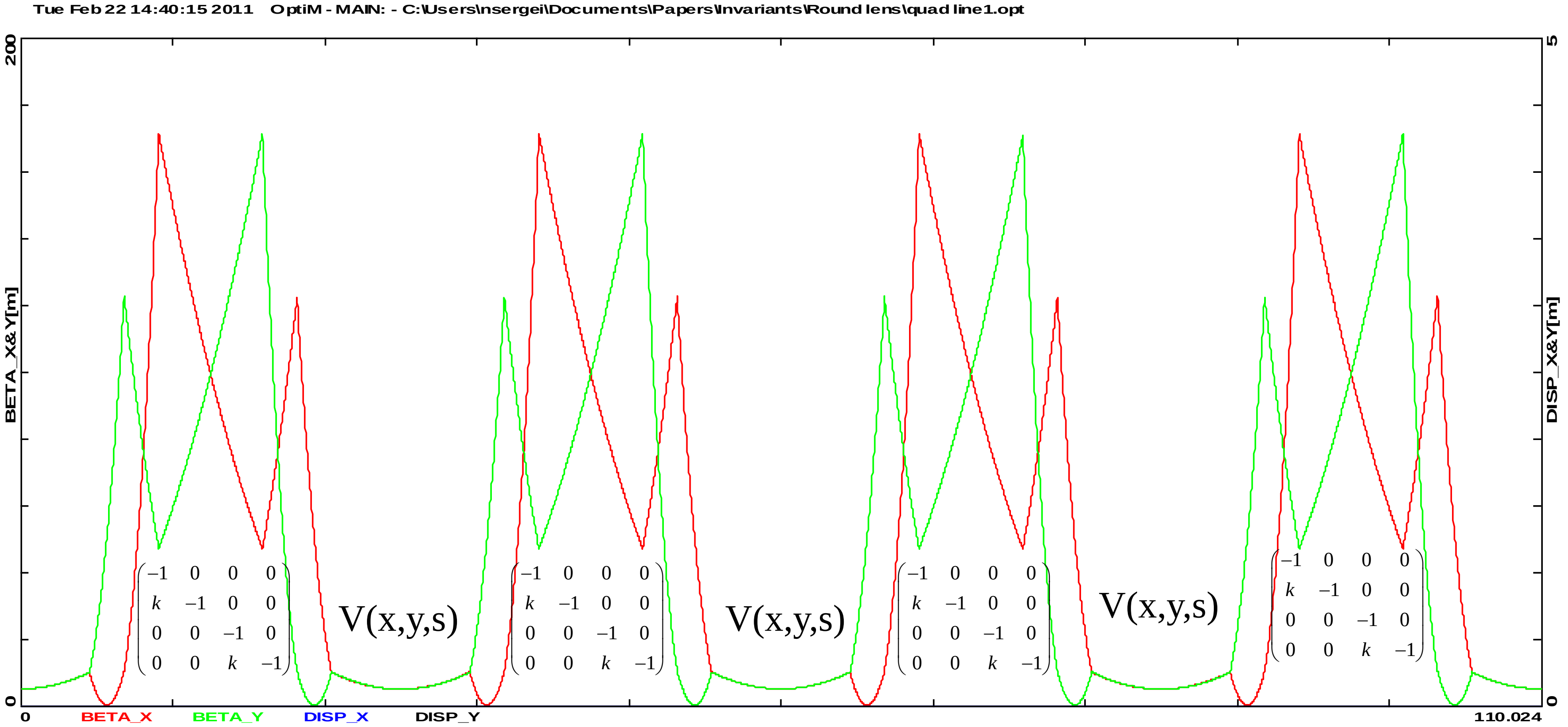
- Could be a solenoid (at low energies), or…

- Could be an optics insert that has a transfer matrix of a thin axially-symmetric focusing lens:



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 \\ k & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & k & -1 \end{pmatrix}$$

Fake thin lens inserts



Main Ideas

1. Start with a time-dependent Hamiltonian:

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi) V\left(x_N \sqrt{\beta(\psi)}, y_N \sqrt{\beta(\psi)}, s(\psi)\right)$$

ψ is a "time" variable

2. Chose the potential to be time-independent in new variables

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + U(x_N, y_N)$$

3. Find potentials *U(x, y)* with the second integral of motion and such that $\Delta U(x, y) = 0$ (for example)

Example 1: quadrupoles

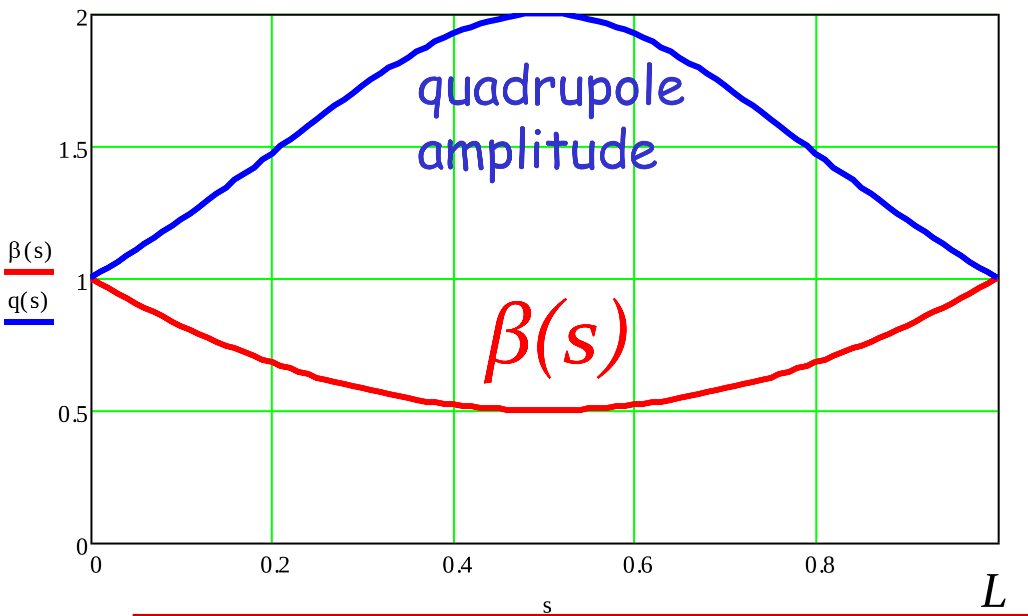
$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi)V\left(x_N\sqrt{\beta(\psi)}, y_N\sqrt{\beta(\psi)}, s(\psi)\right)$$

- This can be made with 10-20 thin quads, each powered independently

$$V(x,y,s) = \frac{q}{\beta(s)^2}\left(x^2 - y^2\right)$$

$$U(x_N, y_N) = q\left(x_N^2 - y_N^2\right)$$

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + q\left(x_N^2 - y_N^2\right)$$



Integrable but still linear…

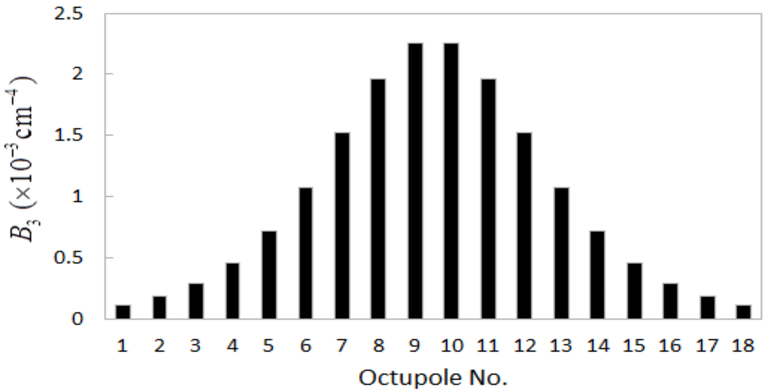
New tunes:

$$\nu_x^2 = \nu_0^2(1+2q)$$
$$\nu_y^2 = \nu_0^2(1-2q)$$

Example 2: Octupoles

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi)V\left(x_N\sqrt{\beta(\psi)}, y_N\sqrt{\beta(\psi)}, s(\psi)\right)$$

- 18 Octupoles



$$V(x,y,s) = \frac{\kappa}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{y^4}{4} - \frac{3x^2y^2}{2}\right)$$

$$U = \kappa\left(\frac{x_N^4}{4} + \frac{y_N^4}{4} - \frac{3y_N^2x_N^2}{2}\right)$$

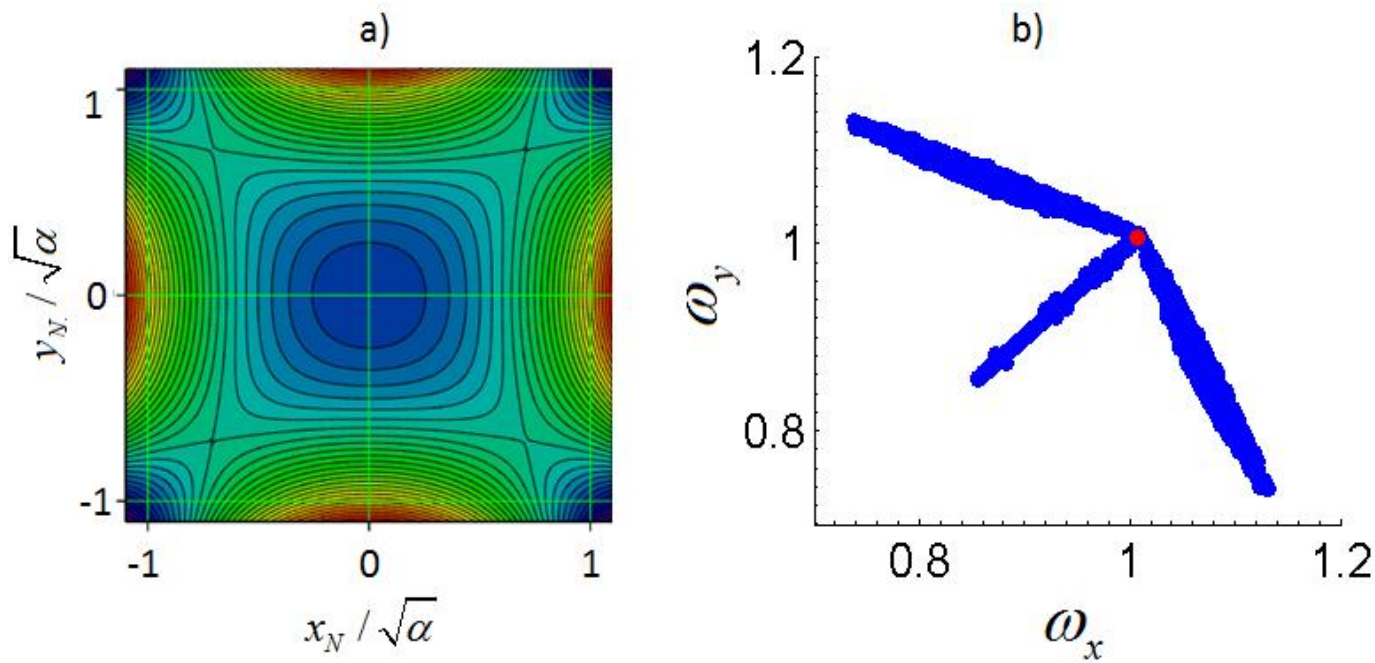
$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{k}{4}\left(x^4 + y^4 - 6x^2y^2\right)$$

This Hamiltonian is NOT integrable, Henon-Heiles – like system

- We will try to make integrable (on Thursday)

Like that: $H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{\alpha}{4}\left(x^4 + y^4\right)$

Example 2: continued



- Contour plot of the octupole potential and the betatron tunes

Example 3: polar coordinates

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi) V\left(x_N \sqrt{\beta(\psi)}, y_N \sqrt{\beta(\psi)}, s(\psi)\right)$$

Obviously, if $V \sim \dfrac{k}{x^2 + y^2}$, the time ($\psi$) dependence would vanish

- Turns out such a potential exists and the system is separable in polar coordinates

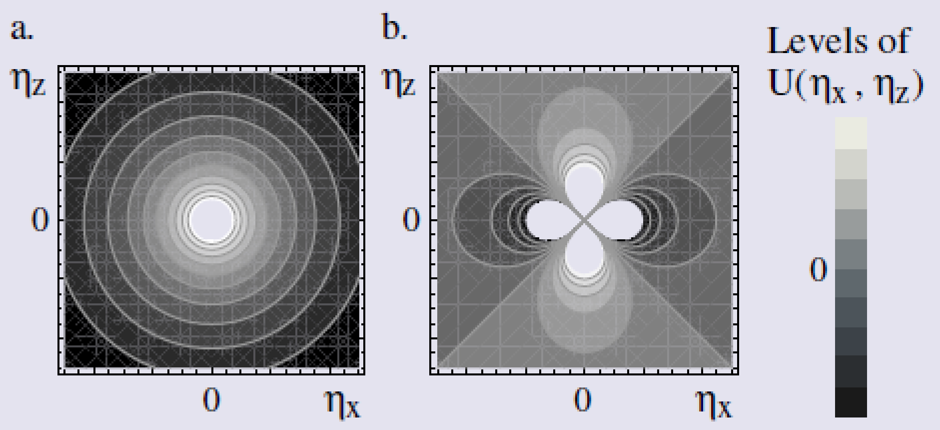
Variables separation in polar coordinates

In polar coordinates the variables separation is possible for the potentials in the form:

$$U(r,\theta) = f(r) + \frac{h(\theta)}{r^2}.$$

Harmonic potentials

- $B \ln r$ — **straight wire carrying a constant current**
- $A \sin(2\theta + \varphi)/r^2$ — **point-like magnetic quadrupole**



Example 3 continued

$$\mathcal{H}[p_r, p_\theta, r, \theta; \psi] = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{r^2}{2} + \frac{A\sin(2\theta + \varphi)}{r^2},$$

with two invariants of motion:

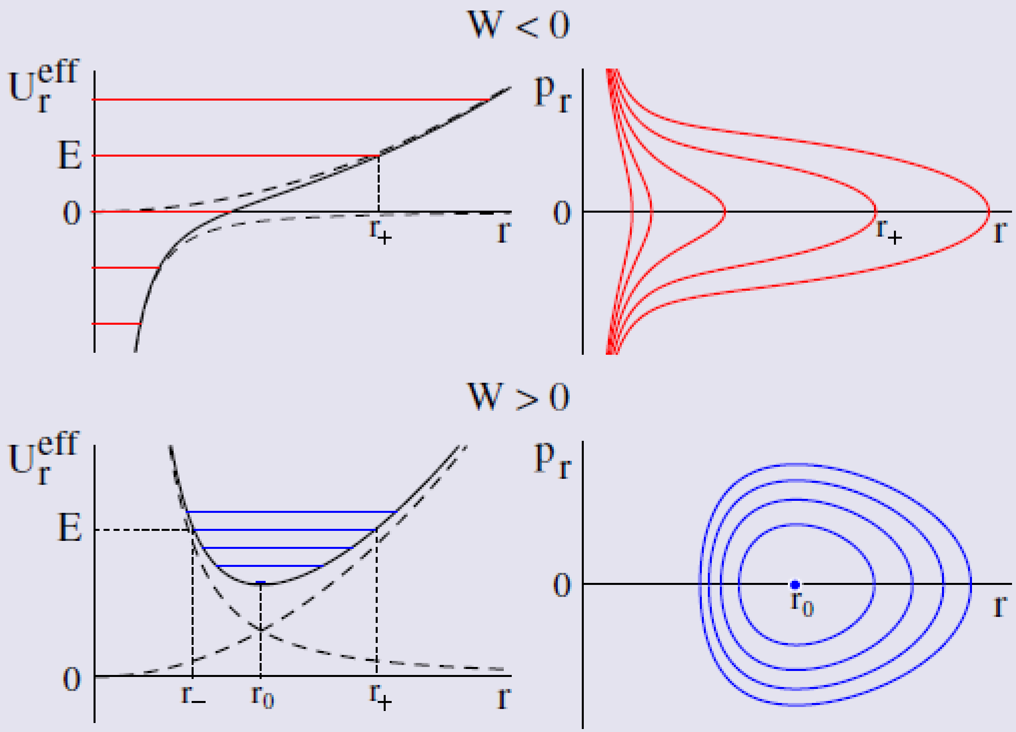
- **energy**

$$\boxed{E = \frac{p_r^2 + r^2}{2} + \frac{W}{r^2}}$$

- **effective angular momentum**

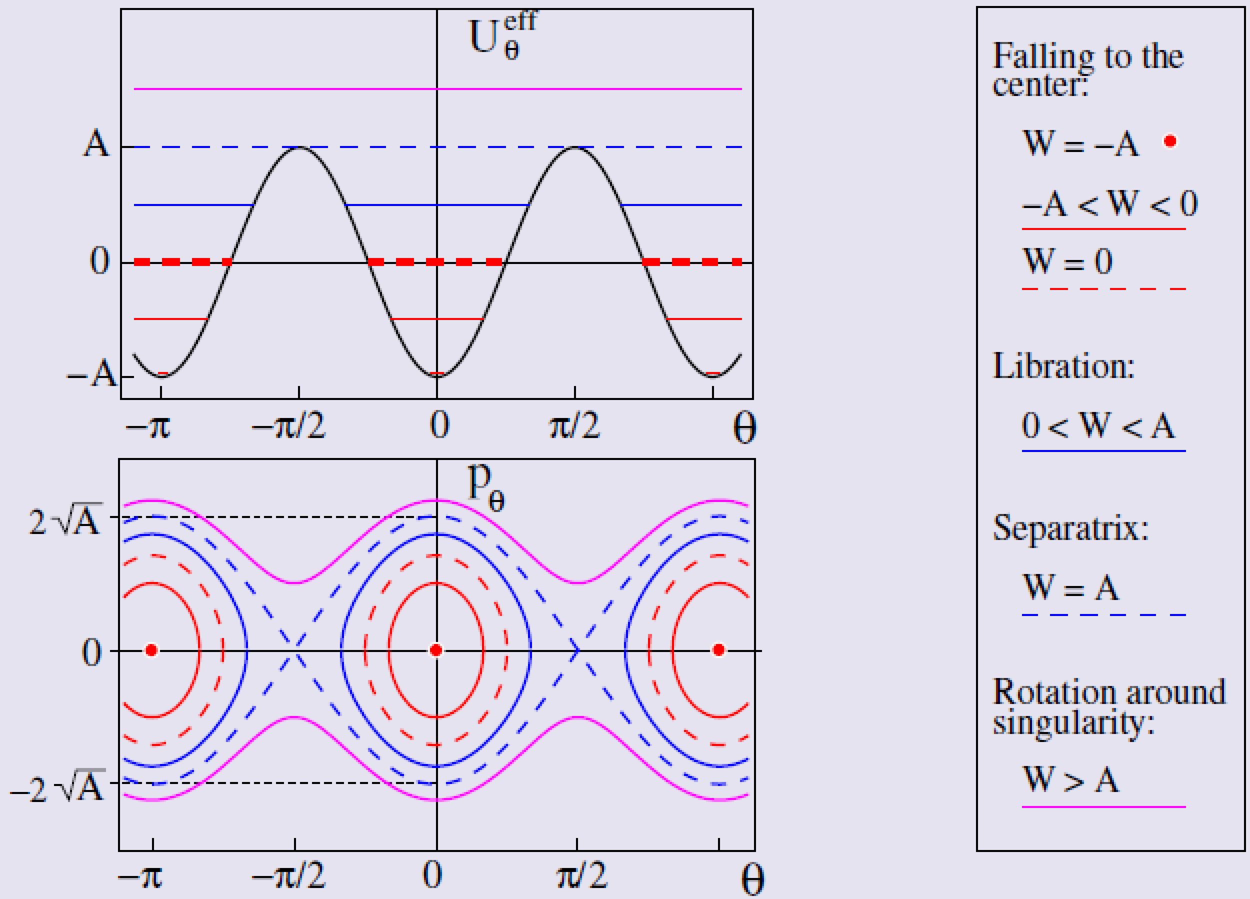
$$\boxed{W = \frac{p_\theta^2}{2} + A\sin(2\theta + \varphi)}$$

Radial motion



$$J_r(E) = \frac{1}{2\pi}\oint p_r\, dr = \frac{E - \sqrt{2W}}{2}, \qquad \omega_r = \frac{\partial \mathcal{H}}{\partial J_r} = 2$$

Angular motion



Classification of trajectories

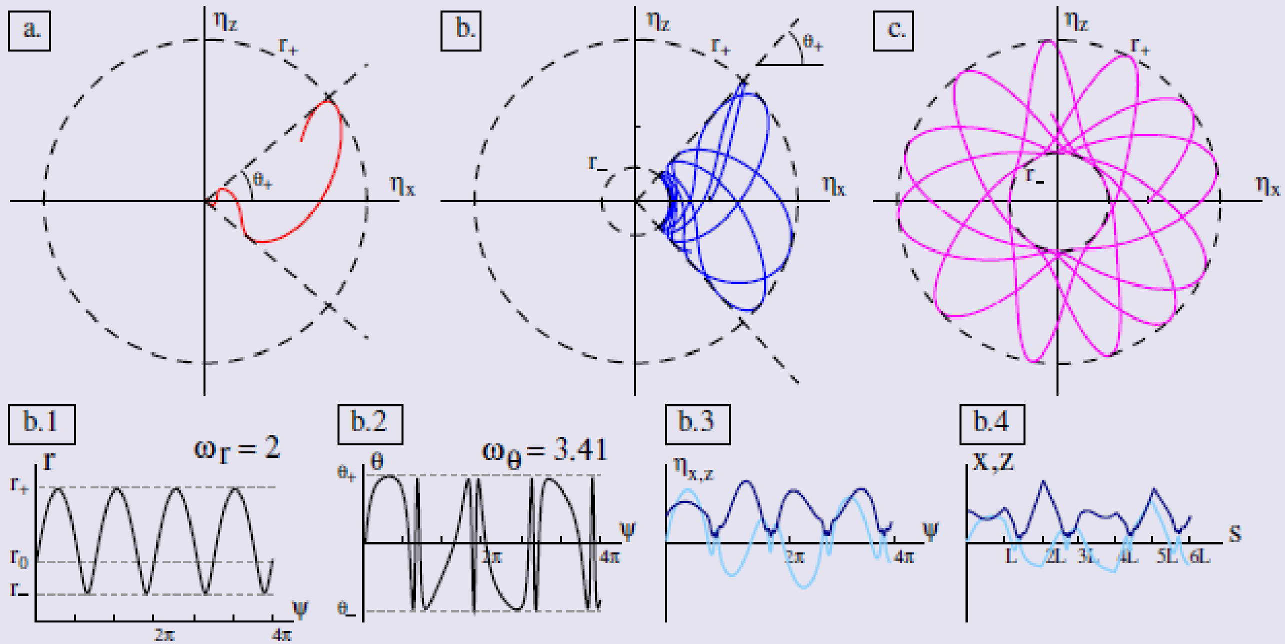


Figure: Particle trajectory in the normalized coordinates for (a.) falling to the center ($-A < W < 0$) (b.) libration ($0 < W < A$) (c.) rotation around the origin ($W > A$).

Summary

- We discussed how to transform the time variable to make the FOFO system "time independent"
- We then discussed several examples of how to exploit this time-independence to make the focusing system nonlinear, yet integrable.