Integrable Particle Dynamics in Accelerators

Lecture 6: "Practical" examples

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Jan 31, 2019

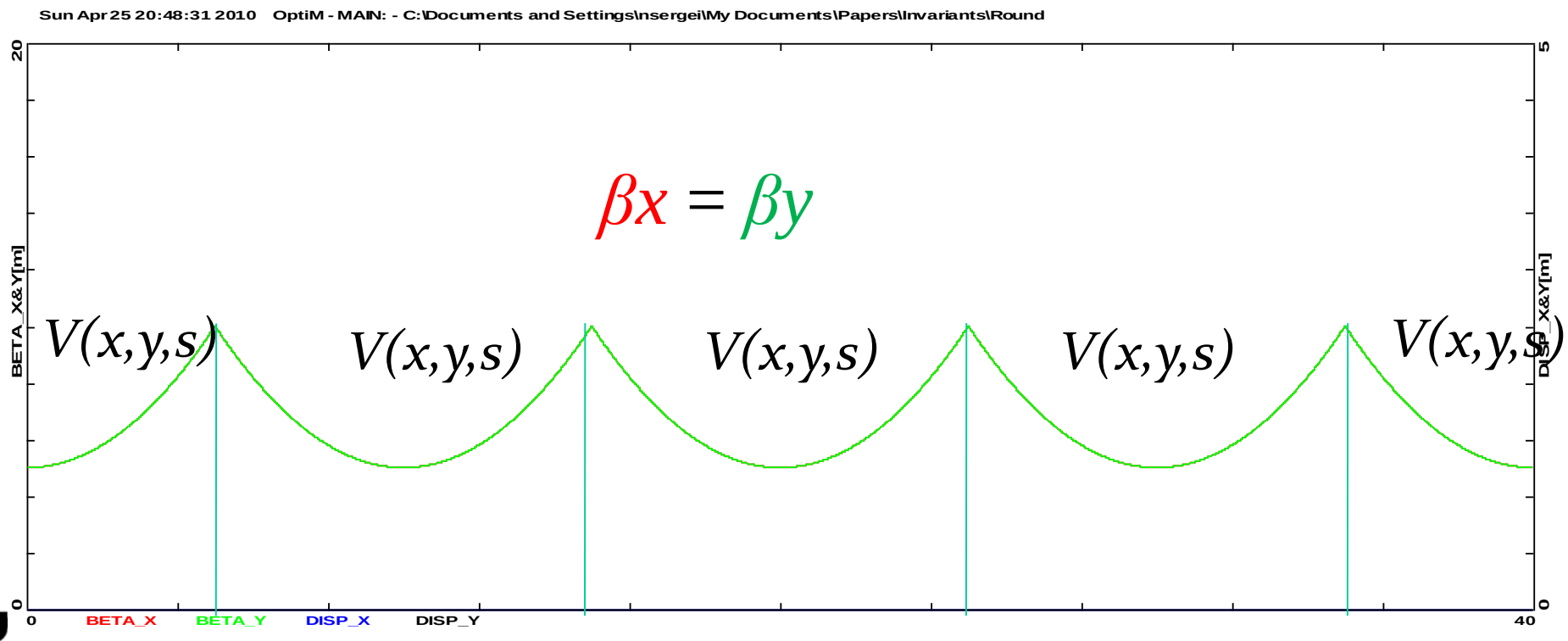
Fermilab

First example

See: Phys. Rev. ST Accel. Beams 13, 084002

Start with a round axially-symmetric LINEAR focusing lattice (FOFO)

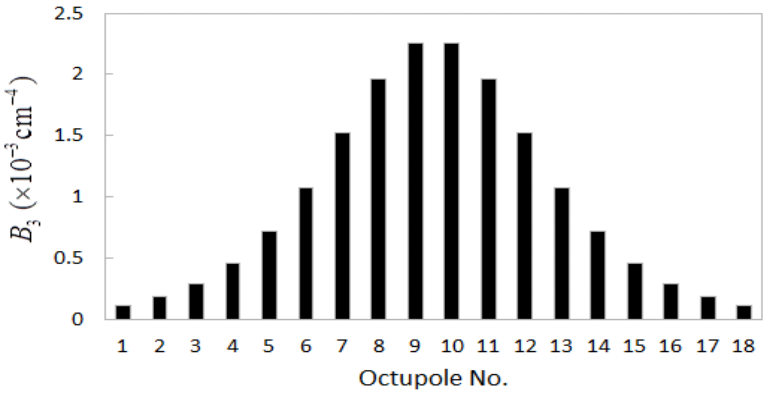
Add a special potential *V(x,y,s)* such that it satisfies either the Laplace or the Poisson equation



Example: Octupoles

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi) V\left(x_N\sqrt{\beta(\psi)}, y_N\sqrt{\beta(\psi)}, s(\psi)\right)$$

- 18 Octupoles



$$V(x,y,s) = \frac{\kappa}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{y^4}{4} - \frac{3x^2y^2}{2}\right)$$

$$U = \kappa\left(\frac{x_N^4}{4} + \frac{y_N^4}{4} - \frac{3y_N^2x_N^2}{2}\right)$$

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{k}{4}\left(x^4 + y^4 - 6x^2y^2\right)$$

This Hamiltonian is NOT integrable, Henon-Heiles – like system

How to uncouple an octupole?

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{\alpha}{4}\left(x^4 + y^4\right)$$

Let's consider a thin octupole

$$\begin{pmatrix} x \\ p_x + q_i\left(x^3 - 3xy^2\right) \\ y \\ p_y + q_i\left(y^3 - 3yx^2\right) \end{pmatrix}$$

- q is octupole's strength

Let's now construct an insert

$$\begin{array}{cc} \text{Oct2} & \text{Oct1} \\ \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & \frac{1}{b_1} & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix} \updownarrow & \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & \frac{1}{a_1} & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & \frac{1}{b_1} \end{pmatrix} \updownarrow \end{array}$$

- Two different octupoles (Oct1 and Oct2), separated by a linear matrix
- The total linear matrix is the identity matrix.
  - So, it's an "invisible insert" from the linear optics stand point.

The resulting transformation

$$\begin{pmatrix} x \\ p_x + x^3\left(q_2 a_1^4 + q_1\right) - 3xy^2\left(q_2 a_1^2 b_1^2 + q_1\right) \\ y \\ p_y + y^3\left(q_2 b_1^4 + q_1\right) - 3x^2 y\left(q_2 a_1^2 b_1^2 + q_1\right) \end{pmatrix}$$

- One can eliminate the coupling terms if

$$q_1 = -q_2 a_1^2 b_1^2$$

The resulting transformation

$$\begin{pmatrix} x \\ p_x + x^3 q_2 a_1^2 \left(a_1^2 - b_1^2\right) \\ y \\ p_y + y^3 q_2 b_1^2 \left(b_1^2 - a_1^2\right) \end{pmatrix}$$

- One can see that the resulting "composite" octupole is of opposite polarity in x and y (albeit uncoupled).
- This is not good because we would like to have focusing in both planes

Let's try a 3-octupole insert

$$\begin{array}{ccc} \text{Oct3} & \text{Oct2} & \text{Oct1} \\ \begin{pmatrix} \frac{1}{a_1 a_2} & 0 & 0 & 0 \\ 0 & a_1 a_2 & 0 & 0 \\ 0 & 0 & \frac{1}{b_1 b_2} & 0 \\ 0 & 0 & 0 & b_1 b_2 \end{pmatrix} & \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & \frac{1}{b_2} \end{pmatrix} & \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & \frac{1}{a_1} & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & \frac{1}{b_1} \end{pmatrix} \end{array}$$

- Again, the linear matrix is the identity matrix, so the insert is "invisible"

We would like to obtain this

$$\begin{pmatrix} x \\ p_x + Qx^3 \\ y \\ p_y + Qy^3 \end{pmatrix}$$

- A thin uncouple "composite" octupole

The final transformation

$$\begin{pmatrix} x \\ p_x + x^3\left(q_3 a_1^4 a_2^4 + q_2 a_1^4 + q_1\right) - 3xy^2\left(q_3 a_1^2 a_2^2 b_1^2 b_2^2 + q_2 a_1^2 b_1^2 + q_1\right) \\ y \\ p_y + y^3\left(q_3 b_1^4 b_2^4 + q_2 b_1^4 + q_1\right) - 3x^2 y\left(q_3 a_1^2 a_2^2 b_1^2 b_2^2 + q_2 a_1^2 b_1^2 + q_1\right) \end{pmatrix}$$

- First, eliminate the coupling terms
- Second, make both planes to be equally "focusing"

Get rid of coupling terms:

$$q_2 = -\frac{q_3 a_1^2 a_2^2 b_1^2 b_2^2 + q_1}{a_1^2 b_1^2}$$

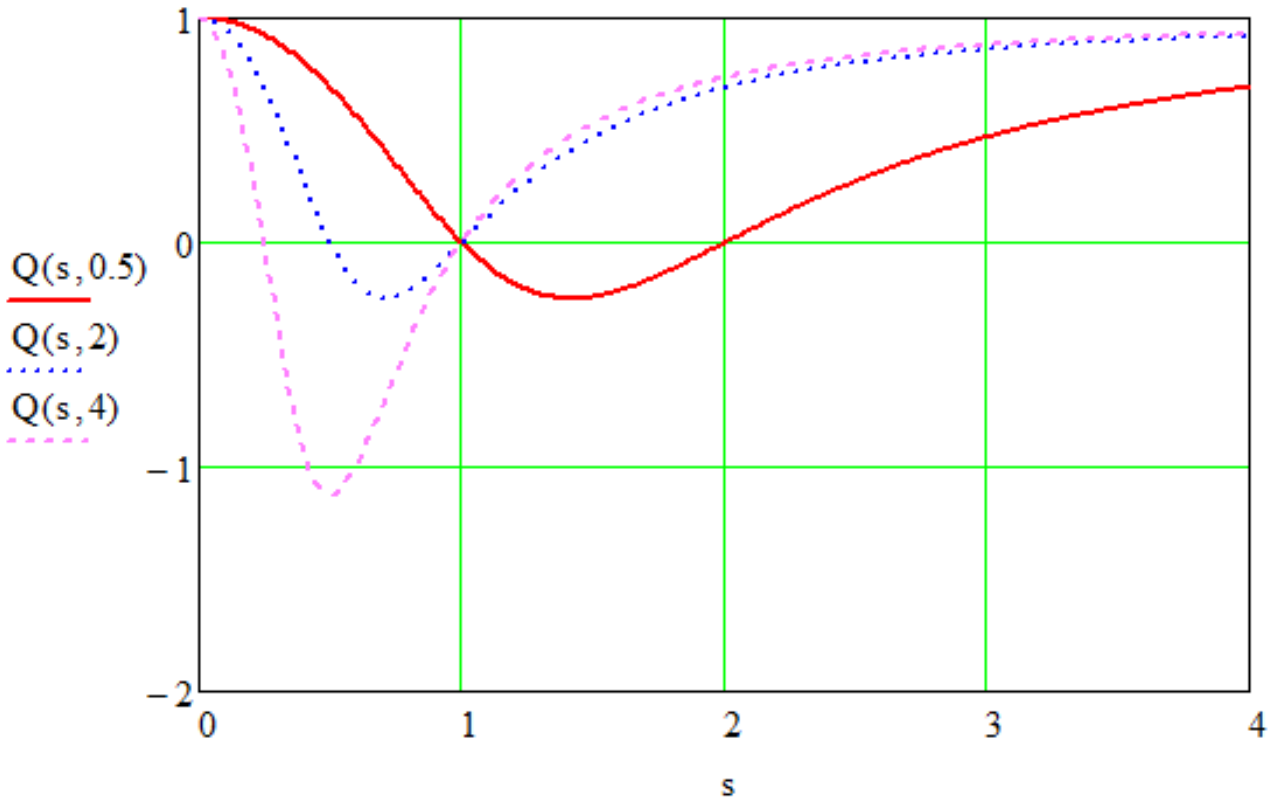
Now make focusing terms equal

$$q_3 = \frac{q_1\left(a_1^4 - b_1^4\right)}{a_1^2 b_1^2\left(a_2^2 - b_2^2\right)\left(a_1^4 a_2^2 + b_1^4 b_2^2\right)}$$

The resulting octupole

$$Q = \frac{q_1\left(s^2-1\right)\left(s^2t^2-1\right)}{s^4t^2+1}$$

- Depends only of ratios $s = \frac{a_1}{b_1}$ and $t = \frac{a_2}{b_2}$



Two other octupoles

$$q2 = -\frac{q1\cdot\left(s^4\cdot t^4 - 1\right)}{a1^2\cdot b1^2\cdot\left(t^2 - 1\right)\cdot\left(s^4\cdot t^2 + 1\right)}$$

$$q3 = \frac{q1\cdot\left(s^4 - 1\right)}{a1^2\cdot b1^2\cdot b2^4\cdot\left(t^2 - 1\right)\cdot\left(s^4\cdot t^2 + 1\right)}$$

- Obviously, *t* and *s* should not be 1.

Example

Example: $q1 := 1$ First octupole

$a1 := 1 \qquad b1 := 2 \qquad a2 := 1 \qquad b2 := 2$

$$s := \frac{a1}{b1} \qquad t := \frac{a2}{b2}$$

$s = 0.5 \qquad t = 0.5 \qquad s \cdot t = 0.25$

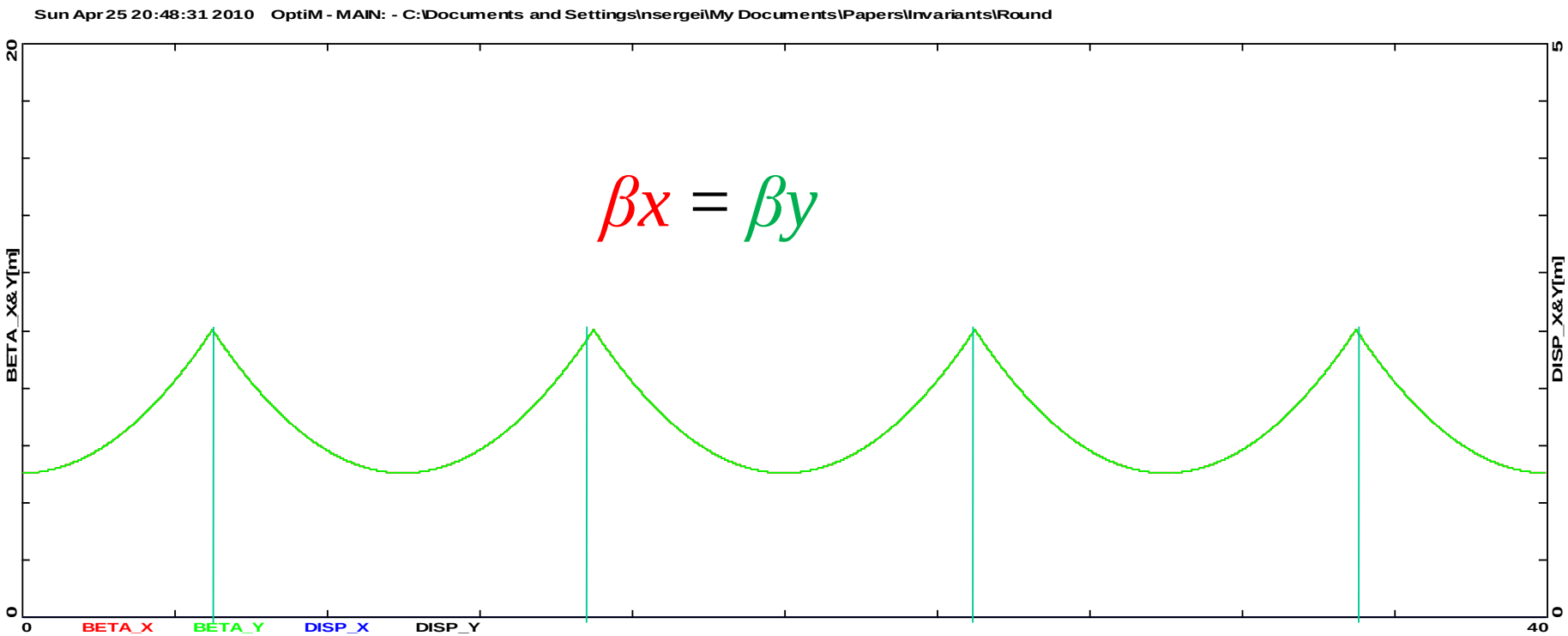
$$q2 := -\frac{q1 \cdot \left(s^4 \cdot t^4 - 1\right)}{a1^2 \cdot b1^2 \cdot \left(t^2 - 1\right) \cdot \left(s^4 \cdot t^2 + 1\right)} \qquad q2 = -0.327$$

$$q3 := \frac{q1 \cdot \left(s^4 - 1\right)}{a1^2 \cdot b1^2 \cdot b2^4 \cdot \left(t^2 - 1\right) \cdot \left(s^4 \cdot t^2 + 1\right)} \qquad q3 = 0.019$$

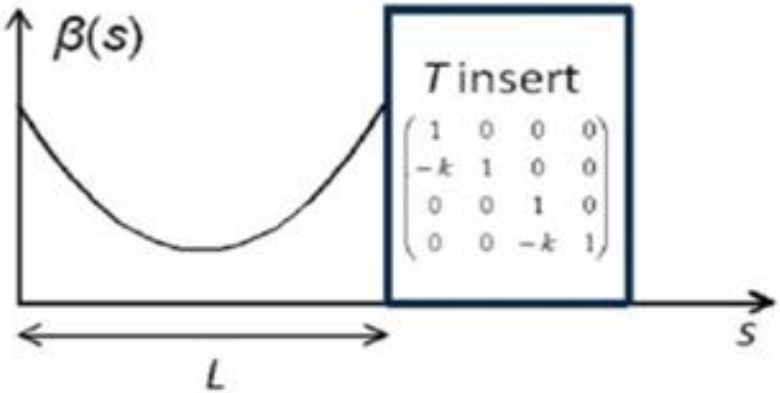
$Q(s, t) = 0.692$

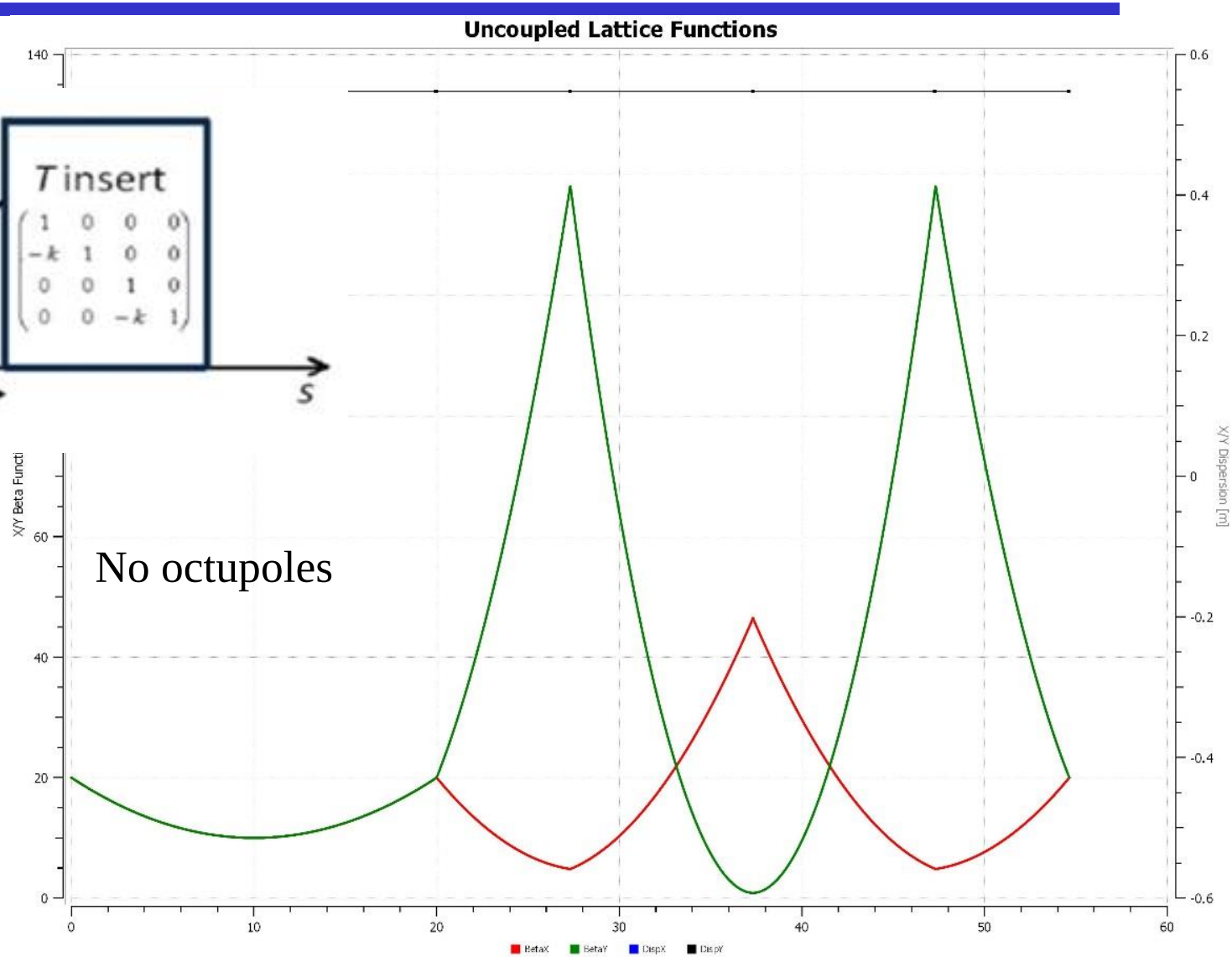
Starting point

Start with a round axially-symmetric LINEAR focusing lattice (FOFO)



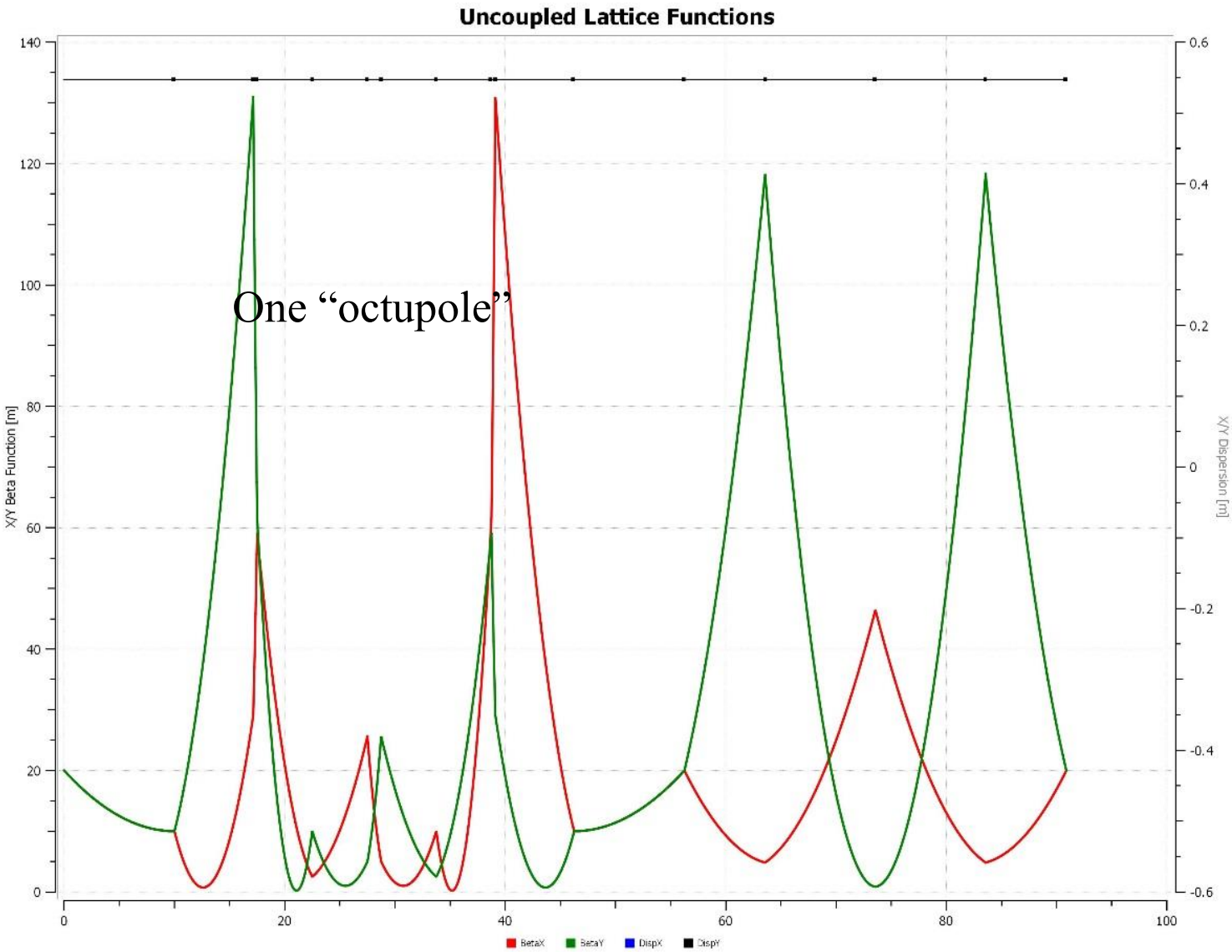
An example with quadrupoles



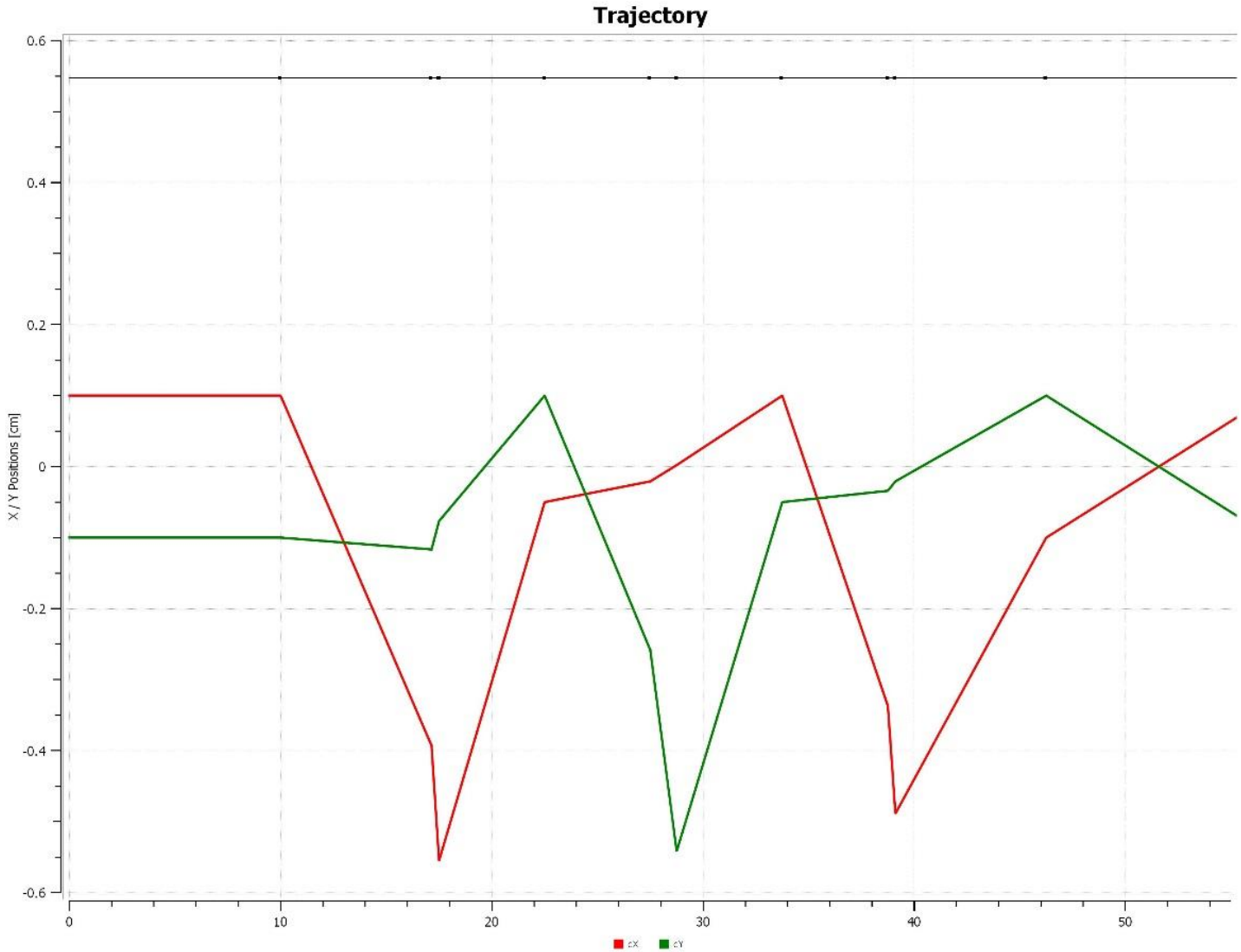


No octupoles

Add one octupole



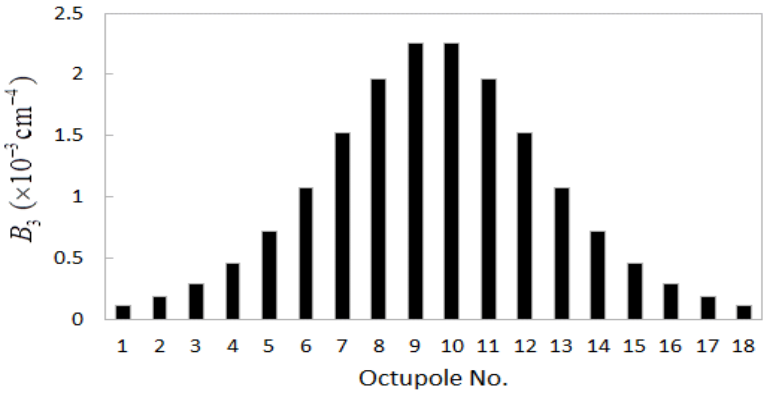
A trajectory through one composite octupole



Now we can construct our octupole insert

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi) V\left(x_N\sqrt{\beta(\psi)}, y_N\sqrt{\beta(\psi)}, s(\psi)\right)$$

- 18 Octupoles



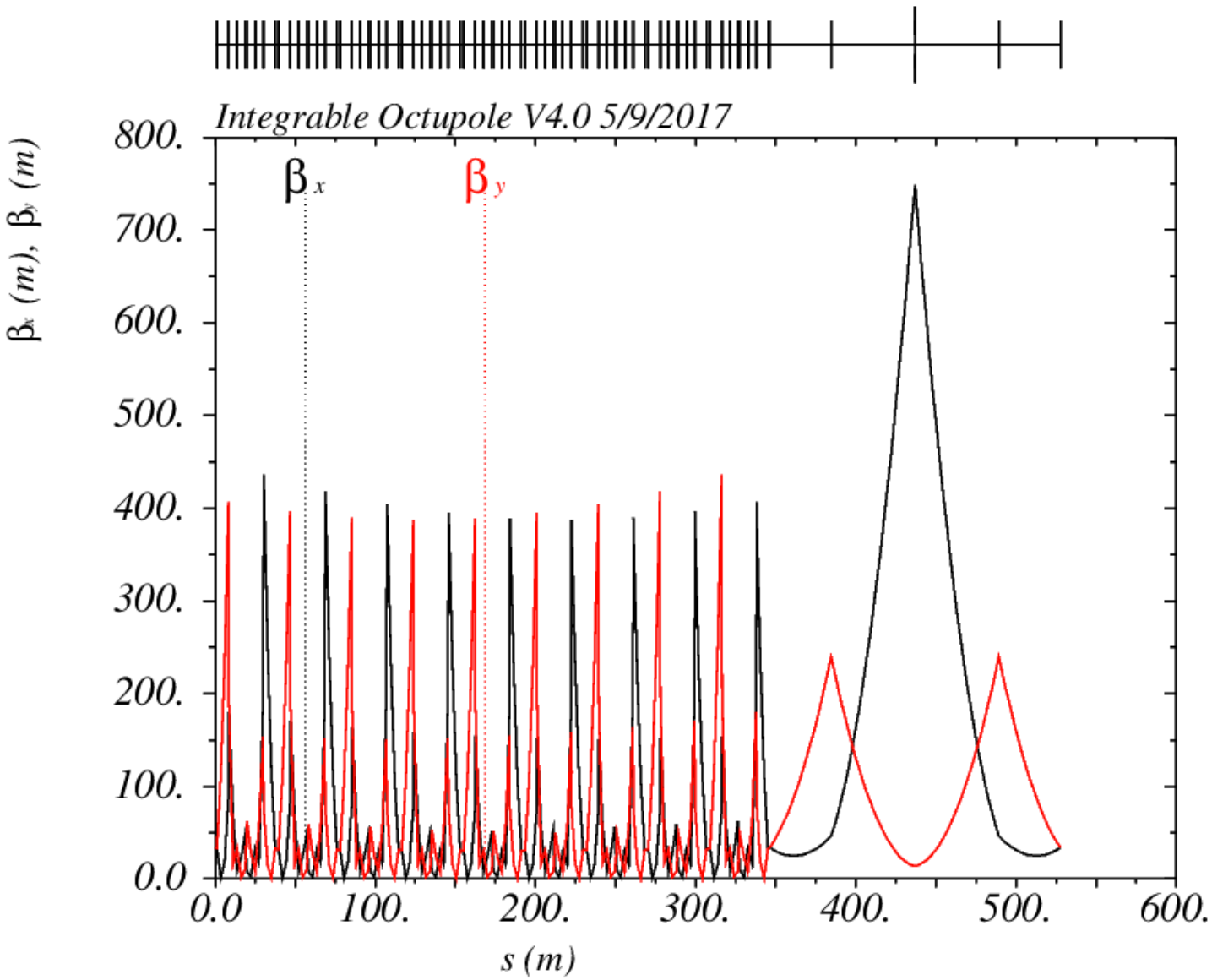
$$V(x,y,s) = \frac{\kappa}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{y^4}{4}\right)$$

$$U = \kappa\left(\frac{x_N^4}{4} + \frac{y_N^4}{4}\right)$$

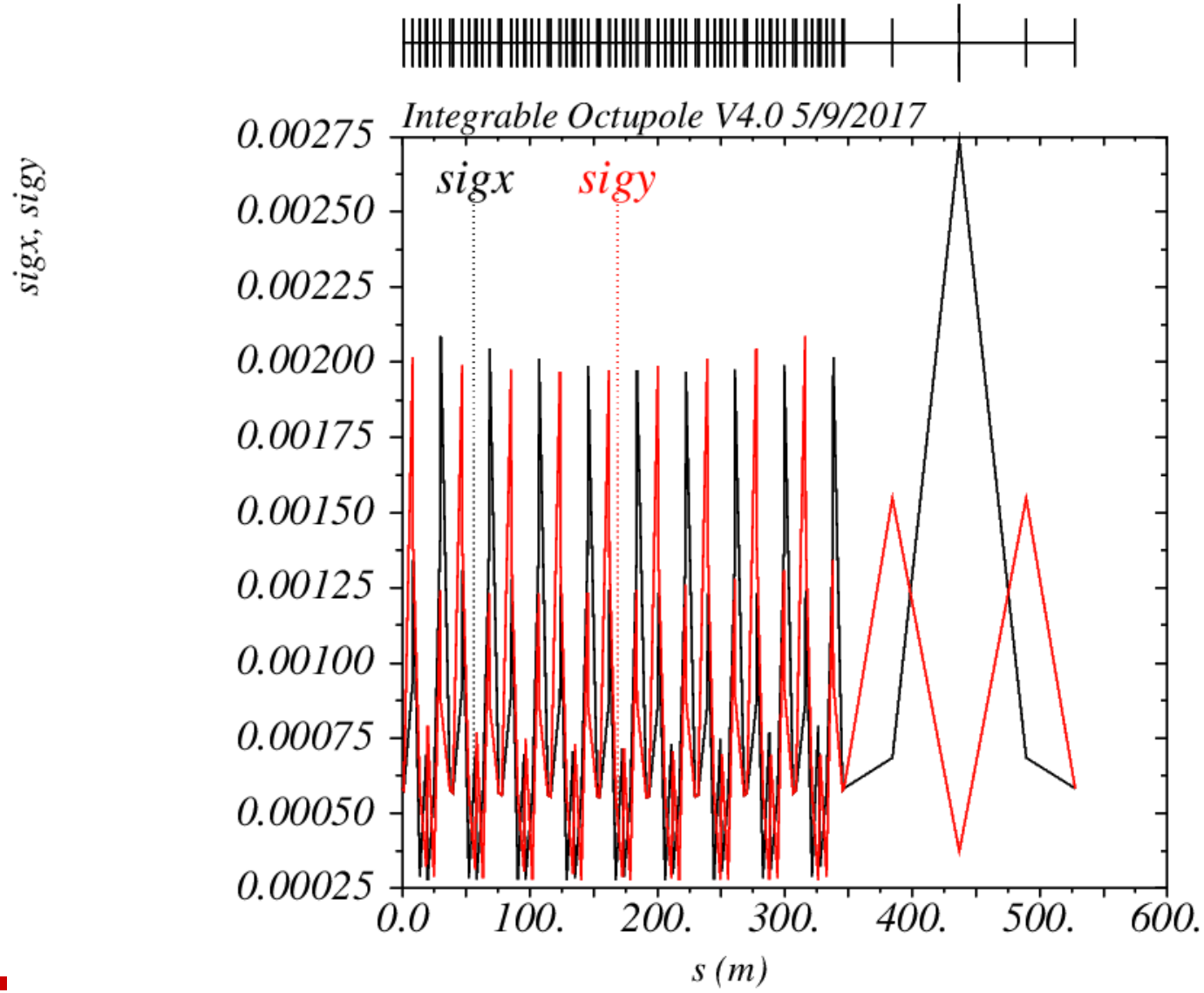
$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{k}{4}\left(x^4 + y^4\right)$$

This Hamiltonian is integrable!

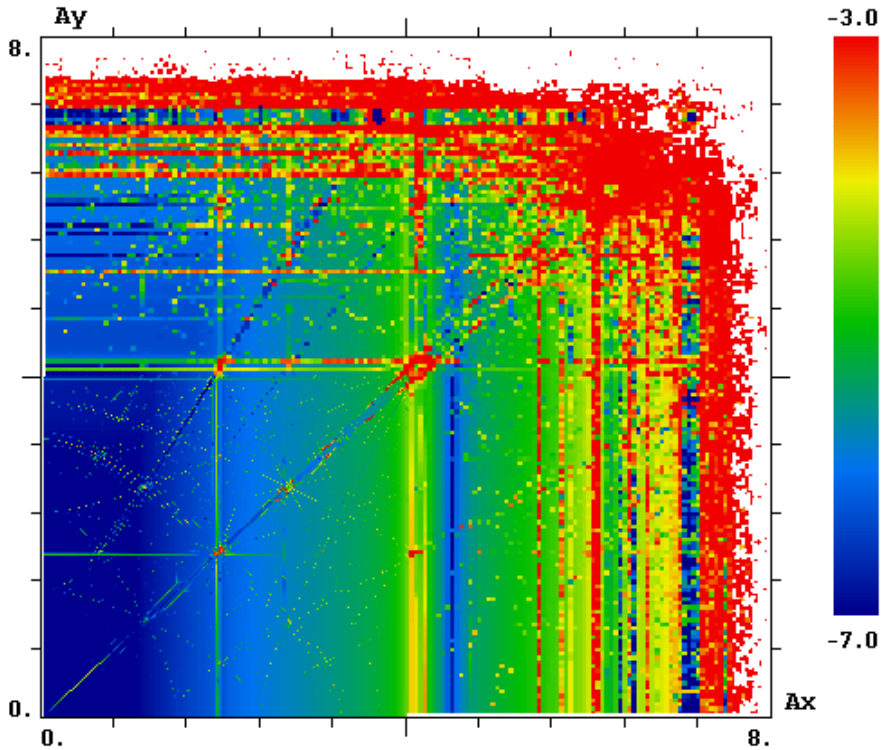
9 composite octupoles. Tunes: 14.1

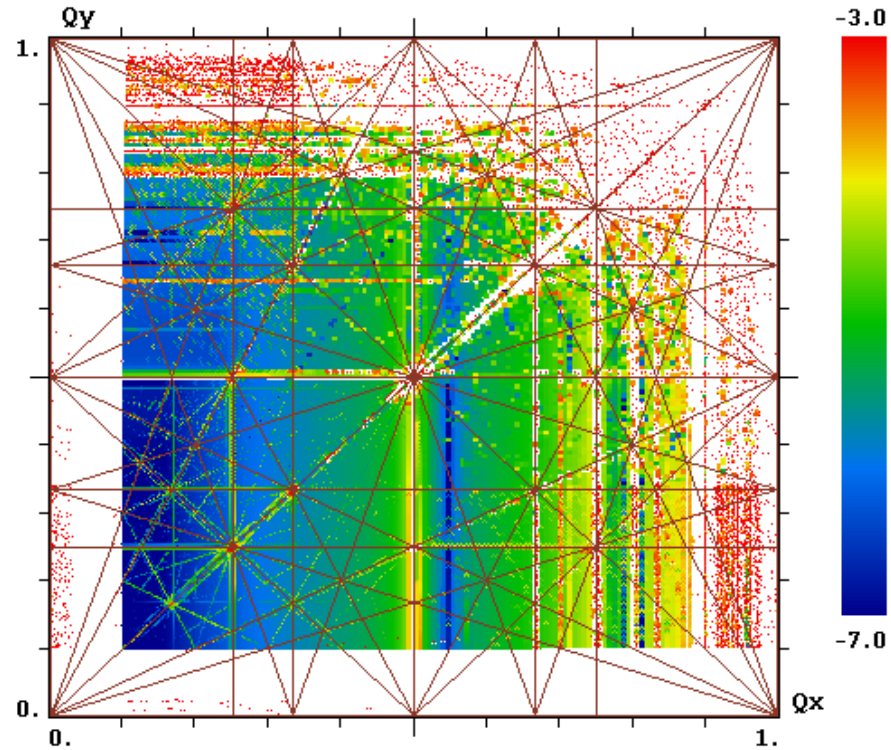


Beam sizes (one sigma)



FMA analysis





Tune spreads of > 0.5 are achievable in simulations!

Other examples

Example: integrable beam-beam

Ideal distribution

$$\lambda(s) = \frac{C}{1+(s/2\beta^*)^2}$$

PAC2013, Paper MOODB1

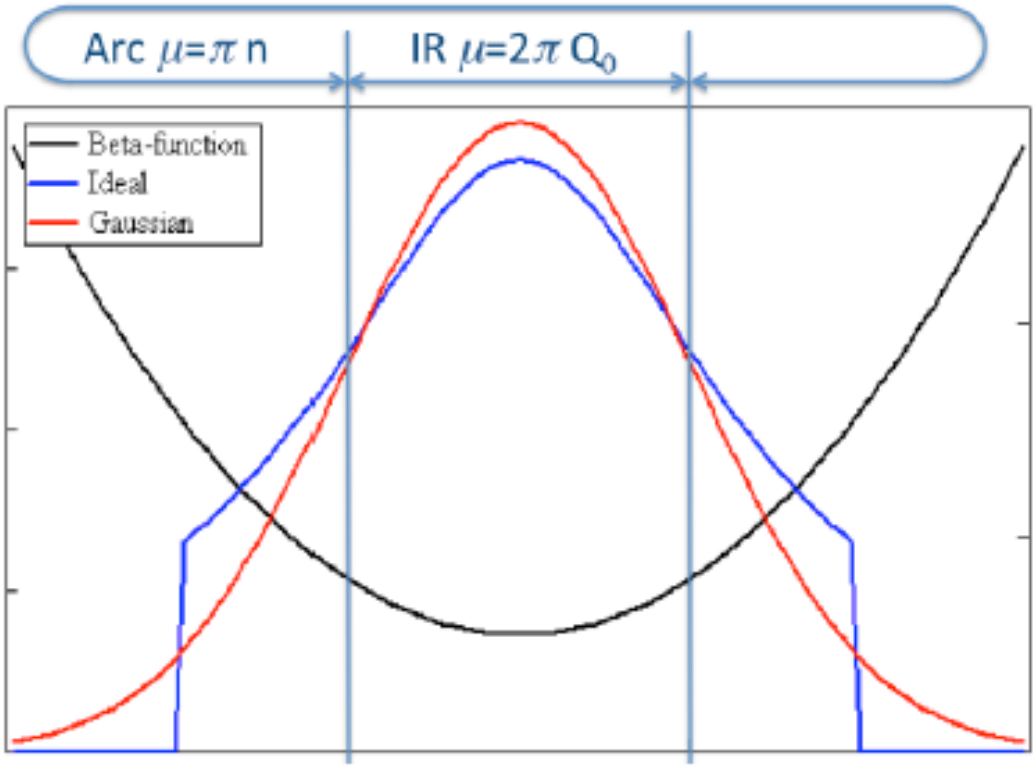


Figure 1: Schematic of the model lattice. Beta-function (black trace), ideal bunch density $\lambda=1/\beta$ (blue) and Gaussian density with $\sigma_z=\sqrt{2}\ \beta^*$ (red) as a function of azimuth. Q_0=0.3. Bunch length for the ideal case is $2\times L$ due to the counter-propagation of two beams.

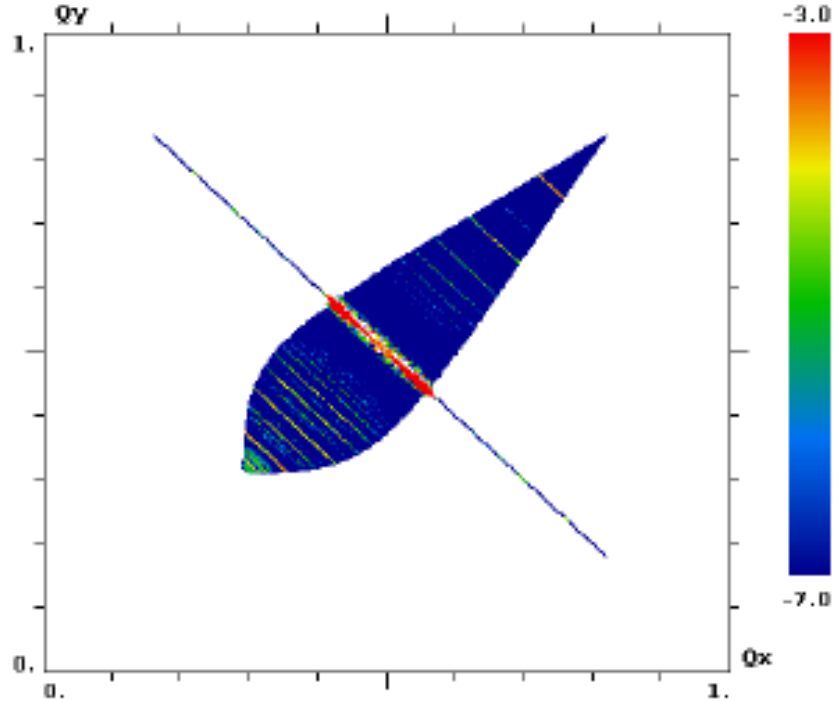


Figure 2: Tune footprint for the case of ideal bunch density $\lambda=1/\beta$, $Q_0=0.3$, and zero synchrotron amplitude. Beam-beam parameter $\xi=1$. Particles with betatron amplitudes up to 20 σ. The color represents tune jitter according to the scale on the right (log scale).

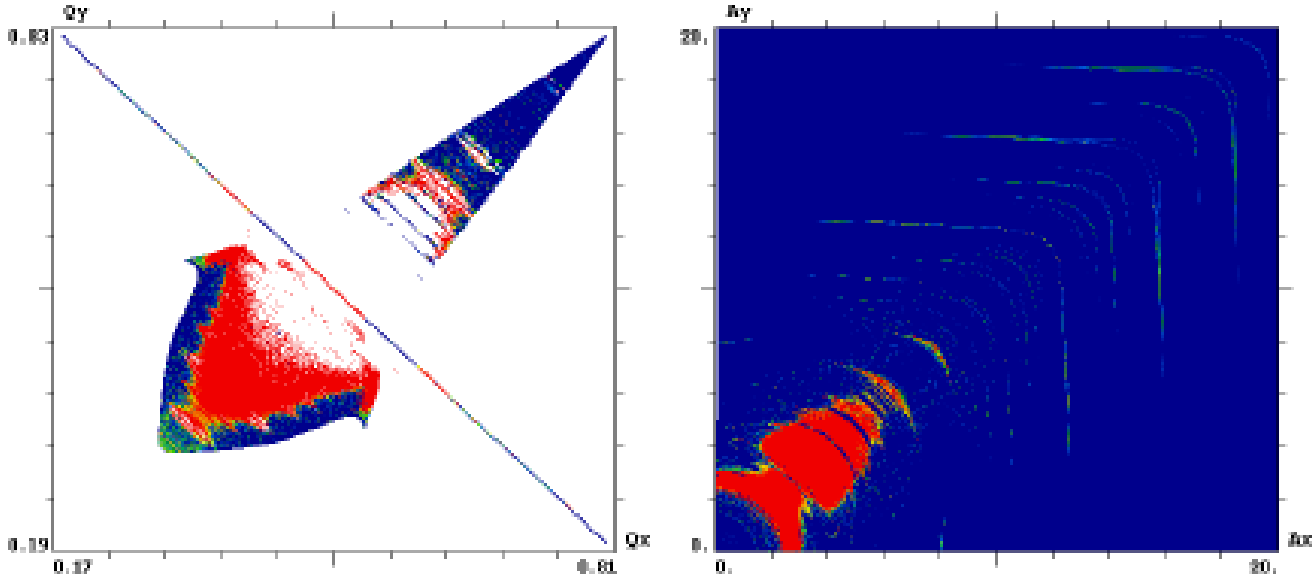
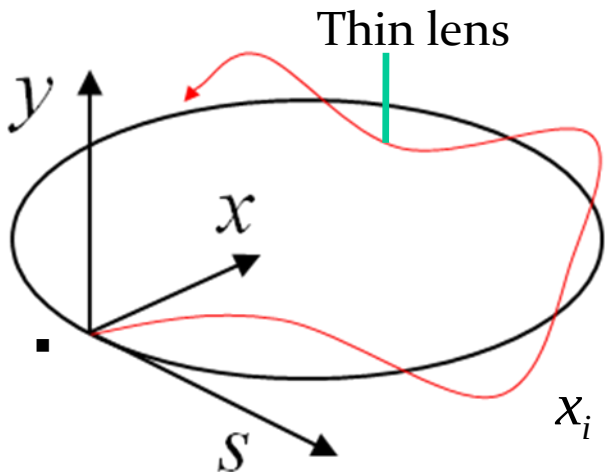


Figure 3: FMA plots for the case of Gaussian bunch density $\sigma_z=\sqrt{2}\ \beta^*$, $Q_0=0.3$, $\xi=1$, and zero synchrotron amplitude. Particles with betatron amplitudes up to 20 σ. Tune footprint (left), FMA in amplitude space (right).

We confirmed that in a specially designed accelerator lattice, and by a careful shaping of the longitudinal profile of strong bunch, the value of beam-beam parameter exceeding one could be achieved for particles with zero synchrotron amplitude. The approach described in this report is advantageous with respect to the previously considered schemes because the machine betatron tune does not need to be close to integer or half-integer. This makes the scheme less sensitive to imperfections of the accelerator lattice.

For particles with non-zero synchrotron amplitude, the integrability is lost but their motion remains stable even at large amplitudes, although a significant emittance growth is induced by the overlapping resonances. Simulations predict that a scheme with the bunch length ratio between the weak and the strong bunch of about 1/10 may be feasible, which could be useful for asymmetric machines such as the electron-ion colliders.

Example: McMillan mapping



SOME THOUGHTS ON STABILITY
IN NONLINEAR PERIODIC FOCUSING SYSTEMS

Edwin M. McMillan

September 5, 1967

1D - thin lens kick $f(x) = -\dfrac{Bx^2 + Dx}{Ax^2 + Bx + C}$

$$x_i = p_{i-1}$$

$$p_i = -x_{i-1} + f(x_i) \qquad Ax^2p^2 + B\left(x^2p + xp^2\right) + C\left(x^2 + p^2\right) + Dxp = \text{const}$$

- 2D – a thin lens solution can be carried over to the 2D case in an axially symmetric system

1. The ring with transfer matrix

$$\begin{pmatrix} cI & sI \\ -sI & cI \end{pmatrix} \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\frac{1}{\beta} & 0 \end{pmatrix}$$

$$c = \cos(\phi)$$

$$s = \sin(\phi)$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. Axially-symmetric thin kick

$$\theta(r) = \frac{kr}{ar^2 + 1}$$

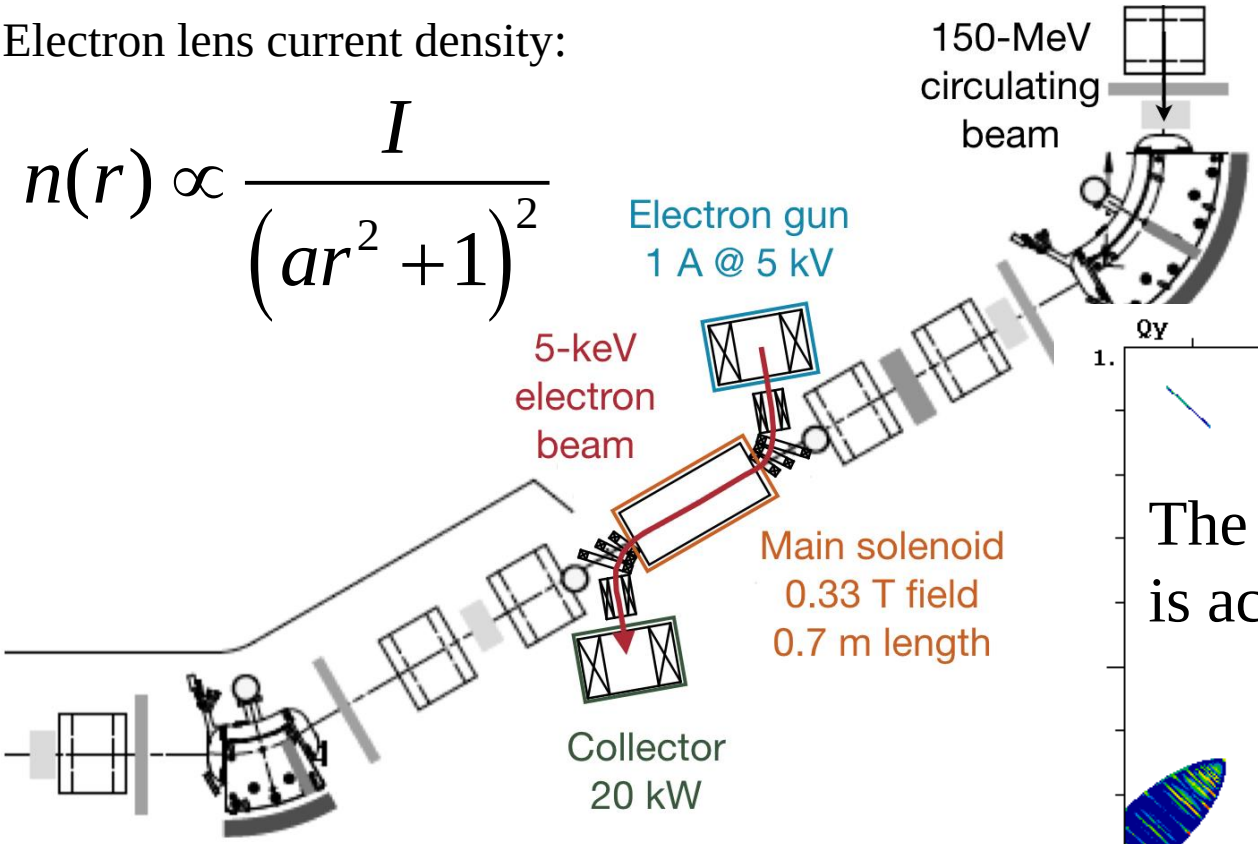
can be created with an electron lens

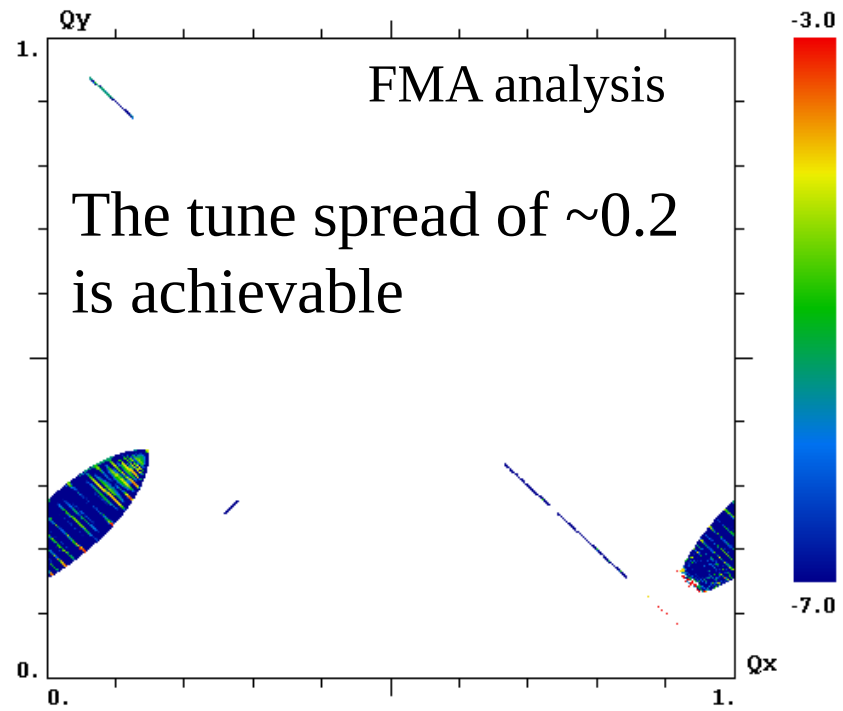
McMillan electron lens

- Capitalize on the Tevatron experience and recent LARP work
- Re-use Tevatron EL components

Electron lens current density:

$$n(r) \propto \frac{I}{\left(ar^2+1\right)^2}$$





G. Stancari

Final thoughts

- This class was more about new ideas than about giving you recipes. Hope you enjoyed it.

- Many more new ideas are still needed!

Accelerator research areas, where integrability would help

- Single particle dynamics:
  1. How to make the dynamical aperture larger? (light sources, colliders)
  2. How to make the tune spread larger? (Landau damping in high-intensity rings)
  3. How to reduce beam halo?
- Multi-particle dynamics:
  1. How reduce detrimental beam-beam effects?
  2. How to compensate space-charge effects?
  3. How to suppress instabilities?
  4. How to reduce beam halo?

Final exam (tomorrow morning)

1. An exotic oscillator (find a small-amplitude oscill. frequency)
2. Linearization of a nonlinear map
3. Poisson brackets
4. Linear map: invariants (integrals), fixed points
5. Propagation of an invariant
6. Fixed points of nonliner maps