# Lecture 3 –4
# or
# 2D Discrete dynamics and McMillan map

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Winter 2017 USPAS Session

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January 20, 2017

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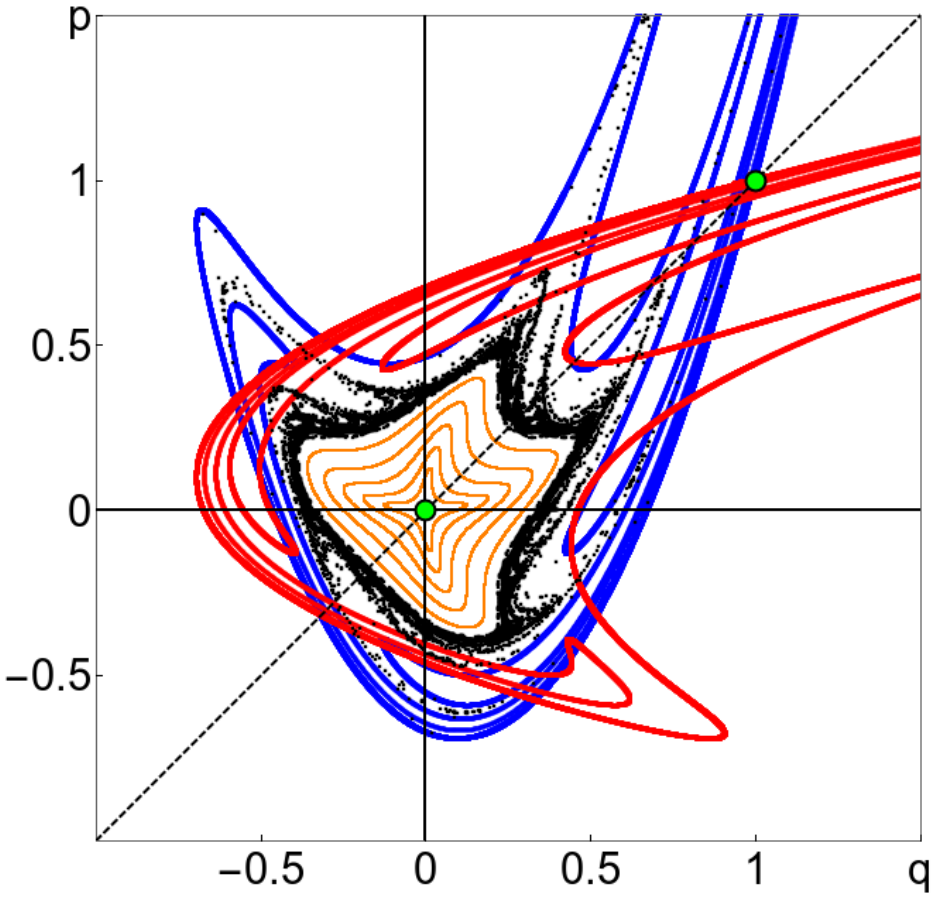
# This presentation is dedicated to Dr. “Slava” Danilov

Henon map: $\begin{bmatrix} q \\ p \end{bmatrix} \to \begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} q \\ p - \lambda\, q^2 \end{bmatrix}$



# 1. McMillan map overview

McMillan map is the planar integrable system with discrete time; it is measure preserving and leaves invariant biquadratic foliation

$$\mathcal{B}(q,p) \equiv \mathbf{Q}\cdot(\mathcal{M}\,\mathbf{P}) = \begin{bmatrix} q^2 \\ q \\ 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathrm{A} & \mathrm{B} & \Gamma \\ \Delta & \mathrm{E} & \Xi \\ \mathrm{K} & \Lambda & \mathcal{K} \end{bmatrix} \begin{bmatrix} p^2 \\ p \\ 1 \end{bmatrix} = 0$$

where the eight coefficients $\mathrm{A}, \mathrm{B}, \ldots, \Lambda$ are parameters of the map, and, $\mathcal{K}$ is an integral for the map which parametrizes a family of non-intersecting curves in the $(q,p)$-plane

$$\boxed{-\mathcal{K} = \mathrm{A}\,q^2p^2 + \mathrm{B}\,q^2p + \Delta\,q\,p^2 + \Gamma\,q^2 + \mathrm{E}\,q\,p + \mathrm{K}\,p^2 + \Xi\,q + \Lambda\,p}\,.$$

# Involutory automorphisms $G_{1,2} = G_{1,2}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$

$$G_{1,2} : (q,p) \mapsto (q',p') \qquad \to \qquad \mathcal{B}(q,p) = \mathcal{B}(q',p')$$

$$G_1 : \quad \begin{aligned} q' &= q \\ p' &= -p - \frac{\mathrm{B}\, q^2 + \mathrm{E}\, q + \Lambda}{\mathrm{A}\, q^2 + \Delta\, q + \mathrm{K}} \end{aligned}$$

$$G_2 : \quad \begin{aligned} q' &= -q - \frac{\Delta\, p^2 + \mathrm{E}\, p + \Xi}{\mathrm{A}\, p^2 + \mathrm{B}\, p + \Gamma} \\ p' &= p \end{aligned}$$

# Asymmetric McMillan map $M_{\mathrm{a}}$

$$M_{\mathrm{a}} = G_1 \circ G_2 \qquad\qquad M_{\mathrm{a}}^{-1} = G_2 \circ G_1$$

$$G_1 \circ G_2 : \qquad \begin{aligned} q' &= -q - \frac{\Delta\, p^2 + \mathrm{E}\, p + \Xi}{\mathrm{A}\, p^2 + \mathrm{B}\, p + \Gamma} \\ p' &= -p - \frac{\mathrm{B}\, q'^2 + \mathrm{E}\, q' + \Lambda}{\mathrm{A}\, q'^2 + \Delta\, q' + \mathrm{K}} \end{aligned}$$

$$G_2 \circ G_1 : \qquad \begin{aligned} q' &= -q - \frac{\Delta\, p'^2 + \mathrm{E}\, p' + \Xi}{\mathrm{A}\, p'^2 + \mathrm{B}\, p' + \Gamma} \\ p' &= -p - \frac{\mathrm{B}\, q^2 + \mathrm{E}\, q + \Lambda}{\mathrm{A}\, q^2 + \Delta\, q + \mathrm{K}} \end{aligned}$$

# Numerical example of $G_{1,2}$, $M_{\mathrm{a}}$

$$\mathrm{A} = 1,\ \mathrm{B} = 2,\ \Gamma = 3,\ \Delta = 4,\ \mathrm{E} = 5,\ \Xi = 6,\ \mathrm{K} = 7,\ \Lambda = 8$$

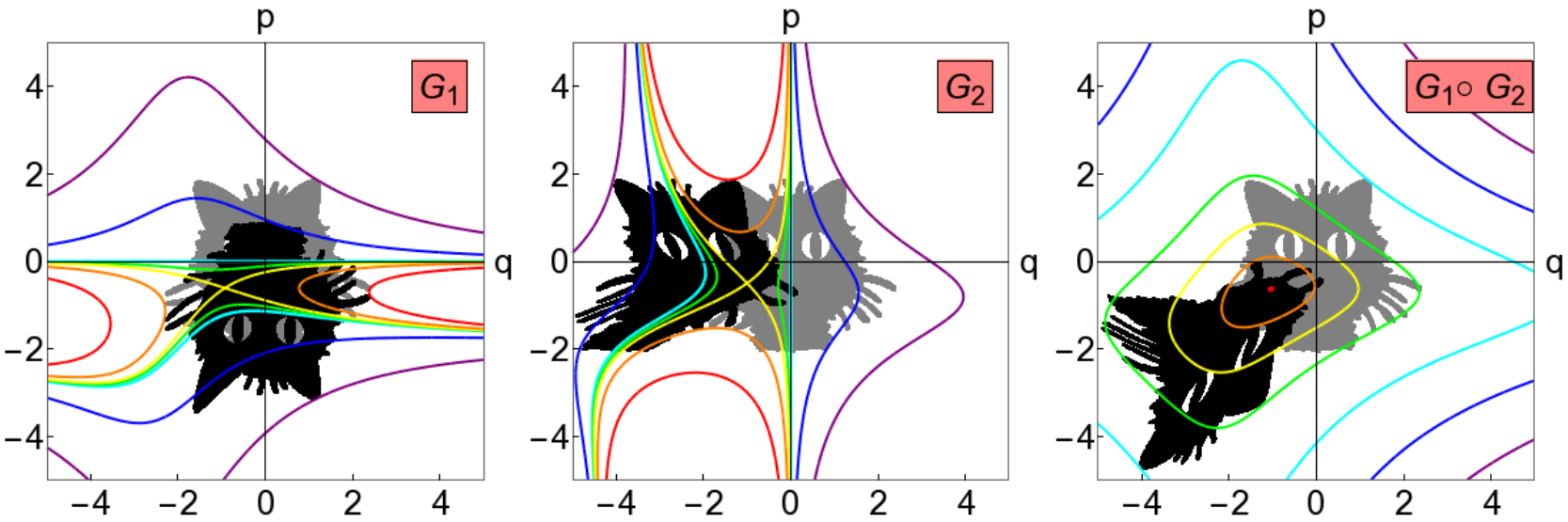



Figure: Picture of a cat (gray) and its image under the application of $G_{1,2}$ and $M_{\mathrm{a}}$ (black). Curves in color show levels of corresponding biquadratic.

# Linear transformations preserving biquadratic $\mathcal{B}(q,p)$

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} : \quad \mathcal{B}(q,p,\mathcal{K}) - \mathcal{B}(q',p',\mathcal{K}) = 0$$

$$\begin{cases} q^4 & : \ (m_{1,1}m_{2,1})^2 \, \mathrm{A} = 0, \\ p^4 & : \ (m_{1,2}m_{2,2})^2 \, \mathrm{A} = 0, \\ q^3 p & : \ 2\, m_{1,1}m_{2,1} \left(m_{1,1}m_{2,2} + m_{1,2}m_{2,1}\right) \mathrm{A} = 0, \\ q\, p^3 & : \ 2\, m_{1,2}m_{2,2} \left(m_{1,1}m_{2,2} + m_{1,2}m_{2,1}\right) \mathrm{A} = 0, \\ q^2 p^2 & : \ \left[(m_{1,1}m_{2,2})^2 + 4\, m_{1,1}m_{1,2}m_{2,1}m_{2,2} + (m_{1,2}m_{2,1})^2 - 1\right] \mathrm{A} = 0, \end{cases} \tag{A1a}$$

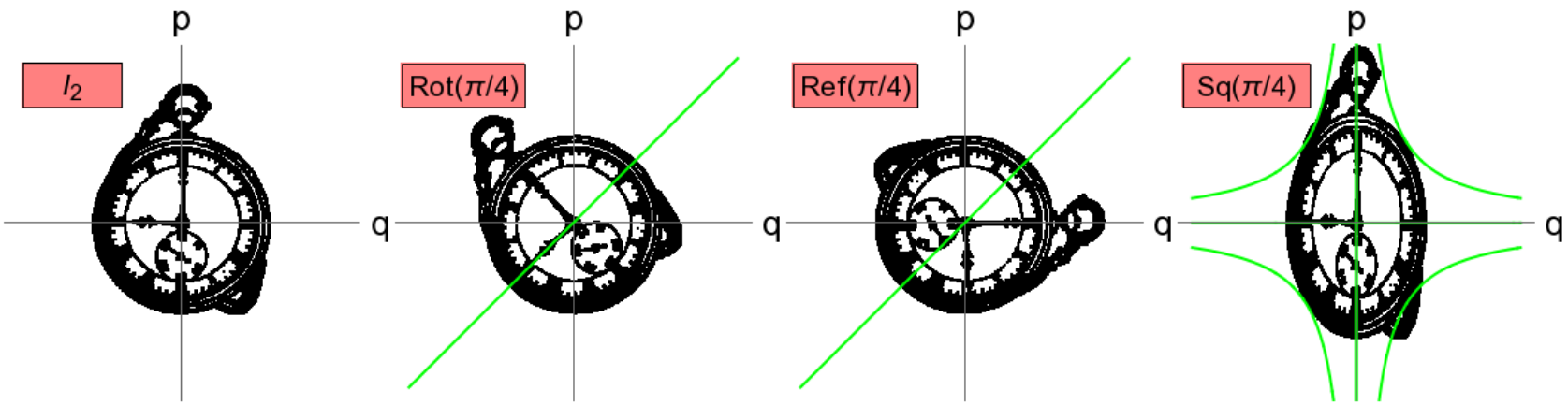
$$\begin{cases} q^3 & : \ m_{1,1}m_{2,1} \left(m_{1,1}\mathrm{B} + m_{2,1}\Delta\right) = 0, \\ p^3 & : \ m_{1,2}m_{2,2} \left(m_{1,2}\mathrm{B} + m_{2,2}\Delta\right) = 0, \\ q^2 p & : \ 2\, m_{1,1}m_{2,1} \left(m_{1,2}\mathrm{B} + m_{2,2}\Delta\right) + m_{1,1}^2 m_{2,2}\mathrm{B} + m_{1,2}m_{2,1}^2 \Delta = \mathrm{B}, \\ q\, p^2 & : \ 2\, m_{1,2}m_{2,2} \left(m_{1,1}\mathrm{B} + m_{2,1}\Delta\right) + m_{1,1}m_{2,2}^2 \Delta + m_{1,2}^2 m_{2,1}\mathrm{B} = \Delta, \end{cases} \tag{A1b}$$

$$\begin{cases} q^2 & : \ \left(m_{1,1}^2 - 1\right) \Gamma + m_{2,1}^2 \mathrm{K} + m_{1,1}m_{2,1}\mathrm{E} = 0, \\ p^2 & : \ m_{1,2}^2 \Gamma + \left(m_{2,2}^2 - 1\right) \mathrm{K} + m_{1,2}m_{2,2}\mathrm{E} = 0, \\ q\, p & : \ 2\left(m_{1,1}m_{1,2}\Gamma + m_{2,1}m_{2,2}\mathrm{K}\right) = \left(1 - m_{1,1}m_{2,2} - m_{1,2}m_{2,1}\right) \mathrm{E}, \end{cases} \tag{A1c}$$

$$\begin{cases} q & : \ (m_{1,1} - 1)\, \Xi + m_{2,1}\Lambda = 0, \\ p & : \ m_{1,2}\Xi + (m_{2,2} - 1)\, \Lambda = 0. \end{cases} \tag{A1d}$$

# Rot, Ref and Sq

$$\mathrm{Rot}: \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Ref}: \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}, \quad \mathrm{Sq}: \begin{bmatrix} \chi & 0 \\ 0 & 1/\chi \end{bmatrix}$$



# Symmetric McMillan map $M_{\mathrm{s}}$

$$M_{\mathrm{s}} = G_1 \circ \mathrm{Ref}\,(\pi/4) = \mathrm{Ref}\,(\pi/4) \circ G_2$$

$$\begin{aligned} q' &= p \\ p' &= -q - \frac{\mathrm{B}\,p^2 + \mathrm{E}\,p + \Xi}{\mathrm{A}\,p^2 + \mathrm{B}\,p + \Gamma} \end{aligned}$$

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$$\mathcal{B}_{\mathrm{s}}(q,p) = \begin{bmatrix} q^2 \\ q \\ 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathrm{A} & \mathrm{B} & \Gamma \\ \mathrm{B} & \mathrm{E} & \Xi \\ \Gamma & \Xi & \mathcal{K} \end{bmatrix} \begin{bmatrix} p^2 \\ p \\ 1 \end{bmatrix} = 0$$

# Squeezed McMillan map $M_{\mathrm{sq}}$

$$M_{\mathrm{sq}} = G_1 \circ \mathrm{Sq}\,(\chi) \circ \mathrm{Ref}\,(\pi/4) = \mathrm{Ref}\,(\pi/4) \circ \mathrm{Sq}\left(\frac{1}{\chi}\right) \circ G_2$$

$$\begin{aligned} q' &= \chi\, p, \\ p' &= \frac{1}{\chi}\left(-q - \frac{\sqrt{\chi}\,\mathrm{B}\,p^2 + \mathrm{E}\,p + \Xi/\sqrt{\chi}}{\mathrm{A}\,p^2 + (\mathrm{B}/\sqrt{\chi})\,p + \Gamma/\chi}\right) \end{aligned}$$

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$$\mathcal{B}_{\mathrm{sq}}(q,p) = \begin{bmatrix} q^2 \\ q \\ 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathrm{A} & \mathrm{B}/\sqrt{\chi} & \Gamma/\chi \\ \sqrt{\chi}\,\mathrm{B} & \mathrm{E} & \Xi/\sqrt{\chi} \\ \chi\,\Gamma & \sqrt{\chi}\,\Xi & \mathcal{K} \end{bmatrix} \begin{bmatrix} p^2 \\ p \\ 1 \end{bmatrix} = 0$$

# 1.2 McMillan map as discrete Hamiltonian structure

For planar systems the area preservation implies the symplectic structure, and thus we can consider the McMillan map as a discrete analog, alternatively a Poincaré map, of integrable Hamiltonian system with one degree of freedom.

$$\det J_{\mathrm{a,s,sq}} = \det \begin{bmatrix} \dfrac{\partial\, q'(q,p)}{\partial\, q} & \dfrac{\partial\, q'(q,p)}{\partial\, p} \\ \dfrac{\partial\, p'(q,p)}{\partial\, q} & \dfrac{\partial\, p'(q,p)}{\partial\, p} \end{bmatrix} = 1$$

# Thin nonlinear McMillan lenses $F_1^{(1,2)}$

$$F_1^{(1)} = G_1 \circ \mathrm{Ref}\,(0) \qquad F_1^{(2)} = \mathrm{Ref}\,(\pi/4) \circ G_2 \circ \mathrm{Ref}\,(\pi/4) \circ \mathrm{Ref}\,(0)$$

$$F_1^{(1)}: \qquad \begin{aligned} q' &= q \\ p' &= p - \frac{\mathrm{B}\,q^2 + \mathrm{E}\,q + \Lambda}{\mathrm{A}\,q^2 + \Delta\,q + \mathrm{K}} \end{aligned}$$

$$F_1^{(2)}: \qquad \begin{aligned} q' &= q \\ p' &= p - \frac{\Delta\,q^2 + \mathrm{E}\,q + \Xi}{\mathrm{A}\,q^2 + \mathrm{B}\,q + \Gamma} \end{aligned}$$

# One-lens cell

$$M_{\mathrm{sq}} = \tilde{F}_1^{(1)} \circ \mathrm{Sq}\,(\chi) \circ \mathrm{Rot}\,(-\pi/2)$$

$$\tilde{F}_1^{(1)}: \qquad \begin{aligned} Q' &= Q \\ P' &= P - \frac{(\mathrm{B}/\sqrt{\chi})q^2 + \mathrm{E}\,q + \sqrt{\chi}\,\Xi}{\mathrm{A}\,q^2 + \sqrt{\chi}\,\mathrm{B}\,q + \chi\,\Gamma} \end{aligned}$$

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$$\mathrm{Sq}\,(\chi) \circ \mathrm{Rot}\,(-\pi/2) = \mathrm{antidiag}\{\chi, -1/\chi\}$$

$$= \begin{bmatrix} \sqrt{\beta_2/\beta_1}\cos(\pi/2) & \sqrt{\beta_1\beta_2}\sin(\pi/2) \\ -\sin(\pi/2)/\sqrt{\beta_1\beta_2} & \sqrt{\beta_1/\beta_2}\cos(\pi/2) \end{bmatrix}$$

where $\chi = \sqrt{\beta_1\beta_2}$

# Normalized coordinates

$$Q \to q = \sqrt{\chi}\, Q \quad \text{and} \quad P \to p = P/\sqrt{\chi}$$

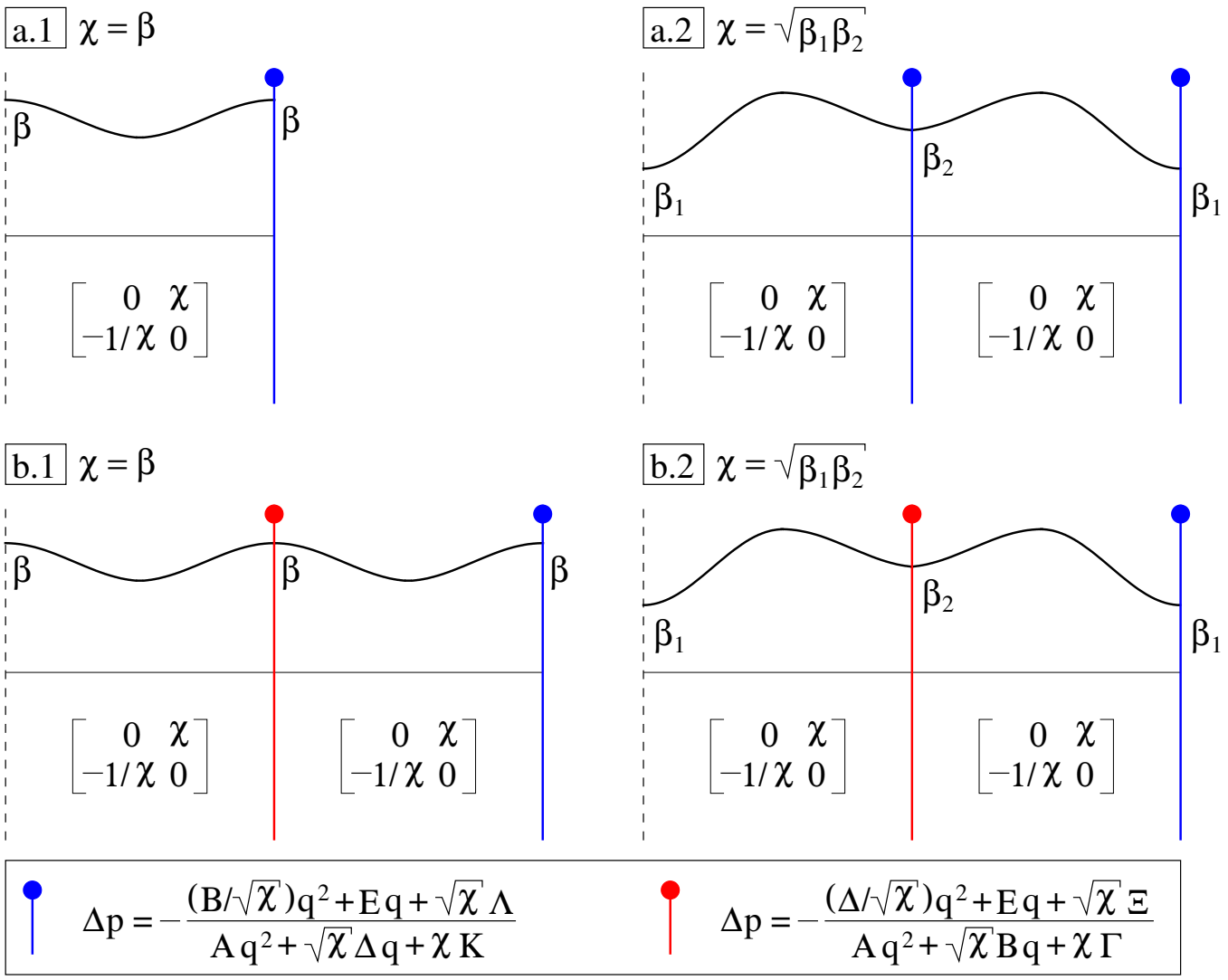
$$\mathcal{B}_{\mathrm{sq}}(Q,P) = \begin{bmatrix} Q^2 \\ Q \\ 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathrm{A} & \mathrm{B} & \Gamma \\ \mathrm{B} & \mathrm{E} & \Xi \\ \Gamma & \Xi & \mathcal{K} \end{bmatrix} \begin{bmatrix} P^2 \\ P \\ 1 \end{bmatrix}.$$

# Two-lens cell

$$M_{\mathrm{a}} = \tilde{F}_1^{(1)} \circ \mathrm{Sq}\,(\chi) \circ \mathrm{Rot}\,(-\pi/2) \circ \tilde{F}_1^{(2)} \circ \mathrm{Sq}\,(\chi) \circ \mathrm{Rot}\,(-\pi/2)$$

$$\tilde{F}_1^{(1)}: \quad \begin{aligned} Q' &= Q \\ P' &= P - \frac{(\mathrm{B}/\sqrt{\chi})q^2 + \mathrm{E}\,q + \sqrt{\chi}\,\Lambda}{\mathrm{A}\,q^2 + \sqrt{\chi}\,\Delta\,q + \chi\,\mathrm{K}} \end{aligned}$$

$$\tilde{F}_1^{(2)}: \quad \begin{aligned} Q' &= Q \\ P' &= P - \frac{(\Delta/\sqrt{\chi})q^2 + \mathrm{E}\,q + \sqrt{\chi}\,\Xi}{\mathrm{A}\,q^2 + \sqrt{\chi}\,\mathrm{B}\,q + \chi\,\Gamma} \end{aligned}$$


a.1 $\chi = \beta$
$\beta$
$\beta$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
a.2 $\chi = \sqrt{\beta_1 \beta_2}$
$\beta_1$
$\beta_2$
$\beta_1$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
b.1 $\chi = \beta$
$\beta$
$\beta$
$\beta$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
b.2 $\chi = \sqrt{\beta_1 \beta_2}$
$\beta_1$
$\beta_2$
$\beta_1$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & \chi \\ -1/\chi & 0 \end{bmatrix}$
$\Delta p = -\frac{(B/\sqrt{\chi})q^2 + Eq + \sqrt{\chi}\,\Lambda}{Aq^2 + \sqrt{\chi}\,\Delta q + \chi K}$
$\Delta p = -\frac{(\Delta/\sqrt{\chi})q^2 + Eq + \sqrt{\chi}\,\Xi}{Aq^2 + \sqrt{\chi}\,Bq + \chi\Gamma}$

# 2.1 Fixed points of $M_{\mathrm{a}}$

**$(q_{\mathrm{f}}, p_{\mathrm{f}}) = M_{\mathrm{a}}(q_{\mathrm{f}}, p_{\mathrm{f}})$**

$$\begin{bmatrix} 2\,q_{\mathrm{f}} \\ 1 \\ 0 \end{bmatrix}^{\mathsf{T}} \mathcal{M} \begin{bmatrix} p_{\mathrm{f}}^2 \\ p_{\mathrm{f}} \\ 1 \end{bmatrix} = 0 \qquad \text{and} \qquad \begin{bmatrix} q_{\mathrm{f}}^2 \\ q_{\mathrm{f}} \\ 1 \end{bmatrix}^{\mathsf{T}} \mathcal{M} \begin{bmatrix} 2\,p_{\mathrm{f}} \\ 1 \\ 0 \end{bmatrix} = 0$$

**All fixed points of $M_{\mathrm{a}}$ are critical points of the $\mathcal{K}(q, p)$**

$$\left.\frac{\partial\,\mathcal{K}(q,p)}{\partial\,q}\right|_{(q_{\mathrm{f}},p_{\mathrm{f}})} = \left.\frac{\partial\,\mathcal{K}(q,p)}{\partial\,p}\right|_{(q_{\mathrm{f}},p_{\mathrm{f}})} = 0$$

# 2.2 Fixed points and 2-cycles of $M_{\mathrm{s}}$

$$(q_{\mathrm{f}}, q_{\mathrm{f}}) = M_{\mathrm{s}}(q_{\mathrm{f}}, q_{\mathrm{f}})$$
$$(q_{(2-\mathrm{c})}, -q_{(2-\mathrm{c})}) = M_{\mathrm{s}} \circ M_{\mathrm{s}}(q_{(2-\mathrm{c})}, -q_{(2-\mathrm{c})})$$

$$\begin{bmatrix} 2\,q_{\mathrm{f}} \\ 1 \\ 0 \end{bmatrix}^{\mathsf{T}} \mathcal{M} \begin{bmatrix} q_{\mathrm{f}}^2 \\ q_{\mathrm{f}} \\ 1 \end{bmatrix} = 0 \qquad \text{and} \qquad \begin{bmatrix} q_{(2-\mathrm{c})}^2 \\ q_{(2-\mathrm{c})} \\ 1 \end{bmatrix}^{\mathsf{T}} \mathcal{M} \begin{bmatrix} -2\,q_{(2-\mathrm{c})} \\ 1 \\ 0 \end{bmatrix} = 0$$

**All fixed points and 2-cycles of $M_{\mathrm{s}}$ are critical points of the $\mathcal{K}(q,p)$**

$$\left.\frac{\partial\,\mathcal{K}(q,p)}{\partial\,q}\right|_{(q_{\mathrm{f}},q_{\mathrm{f}})} = \left.\frac{\partial\,\mathcal{K}(q,p)}{\partial\,p}\right|_{(q_{(2-\mathrm{c})},-q_{(2-\mathrm{c})})} = 0$$

# 2.3 Stability of fixed points

**Fixed point ($n$-cycle) is stable if $|\mathrm{Tr}\, J(M)| < 2$**

Asymmetric McMillan map, $M_{\mathrm{a}}$:

$$4\left(\mathrm{A}\, q_{\mathrm{f}}^2 + \Delta\, q_{\mathrm{f}} + \mathrm{K}\right)\left(\mathrm{A}\, p_{\mathrm{f}}^2 + \mathrm{B}\, p_{\mathrm{f}} + \Gamma\right) - \\ - \left[4\,\mathrm{A}\, q_{\mathrm{f}}\, p_{\mathrm{f}} + 2\left(\mathrm{B}\, q_{\mathrm{f}} + \Delta\, p_{\mathrm{f}}\right) + \mathrm{E}\right]^2 > 0$$

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Symmetric McMillan map, $M_{\mathrm{s}}$:

- $|2(\mathrm{A}\, q_{\mathrm{f}}^2 + \mathrm{B}\, q_{\mathrm{f}} + \Gamma)| - |4 q_{\mathrm{f}}\left(\mathrm{A}\, q_{\mathrm{f}} + \mathrm{B}\right) + \mathrm{E}| > 0$
- $4[(\mathrm{A}\, q_{(2-\mathrm{c})}^2 + \Gamma)^2 - \mathrm{B}^2\, q_{(2-\mathrm{c})}^2] - (\mathrm{E} - 4\,\mathrm{A}\, q_{(2-\mathrm{c})}^2)^2 > 0$

# 2.4 Normal forms

**A. Iatrou and J. Roberts** *Integrable mappings of the plane preserving biquadratic invariant curves II*

Using an appropriate homographic (modular) transformation

$$\bar{x} = f(x) = \frac{a\,x + b}{c\,x + d} : \; a,\, b,\, c,\, d \in \mathbb{R},\; a\,d - b\,c \neq 0$$

any particular asymmetric or symmetric biquadratic can be reduced to normal form.

# 3. Canonical McMillan map $M_{\mathrm{c}}$

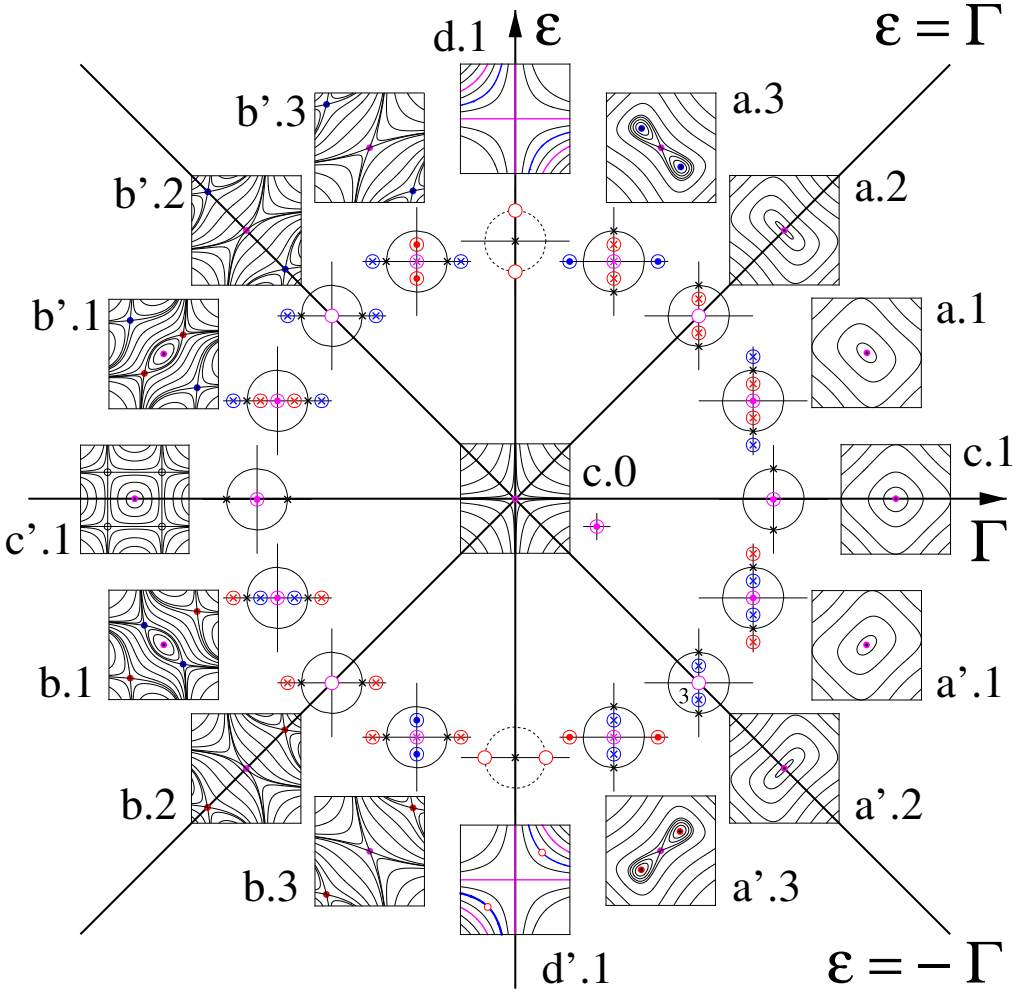
## $\mathrm{A} = 1,\ \mathrm{B} = \Xi = 0,\ \mathrm{E} = 2\,\epsilon$

$$M_{\mathrm{c}}: \qquad \begin{aligned} q' &= \ \ p \\ p' &= -q - \frac{2\,\epsilon\,p}{p^2 + \Gamma} \end{aligned}$$

## Canonical biquadratic

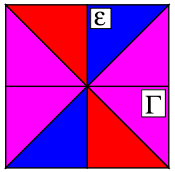
$$\begin{aligned} \mathcal{B}_{\mathrm{c}}(q, p, \mathcal{K}) &\equiv q^2 p^2 + \Gamma\left(q^2 + p^2\right) + 2\,\epsilon\,q\,p + \mathcal{K} \\ &= \begin{bmatrix} q^2 \\ q \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \Gamma \\ 0 & 2\,\epsilon & 0 \\ \Gamma & 0 & \mathcal{K} \end{bmatrix} \begin{bmatrix} p^2 \\ p \\ 1 \end{bmatrix} = 0 \end{aligned}$$

# 3.1 Bifurcation diagram for canonical McMillan map




| Fixed points and 2−cycles | center | saddle | neutral |
|---|---|---|---|
| $f_1 = 0$ | ⊙ | ⊗ | ○ |
| $f_{2,3} = \pm\sqrt{-\varepsilon-\Gamma}$ | ⊙ | ⊗ | ○ |
| $c_{1,2} = \pm\sqrt{\varepsilon-\Gamma}$ | ⊙ | ⊗ | ○ |

Stability




| Critical points of K(q,p) | extrema | saddle |
|---|---|---|
| $K = 0$ | • | ⊕ |
| $K = (\varepsilon+\Gamma)^2$ | • | ⊕ |
| $K = (\varepsilon-\Gamma)^2$ | • | ⊕ |
| $K = \Gamma^2$ | | ⊕ |

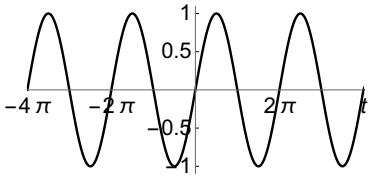
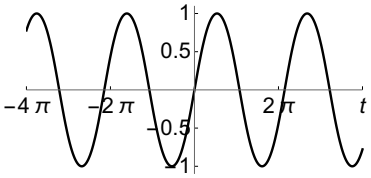
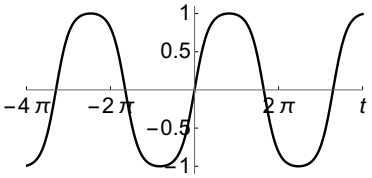
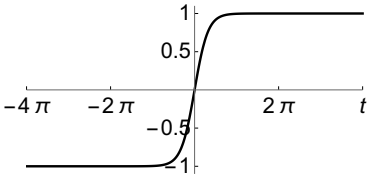
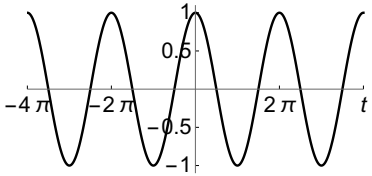
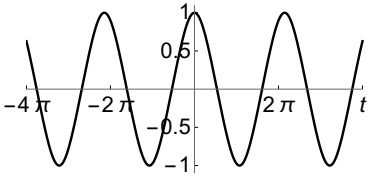
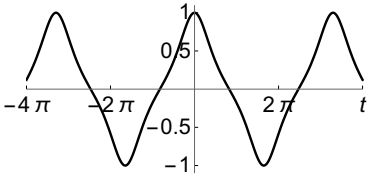
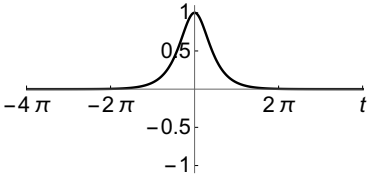
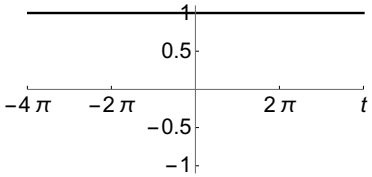
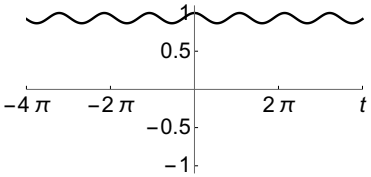
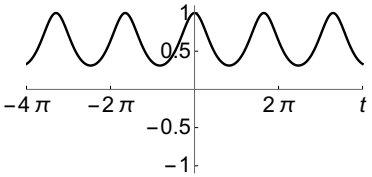
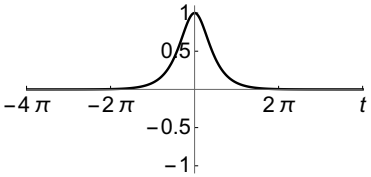
# 3.2 Parametrization of individual curves

$q, p \to q, p/\sqrt[4]{|\mathcal{K}|}$ $\qquad$ $\epsilon, \Gamma \to \epsilon, \Gamma/\sqrt{|\mathcal{K}|}$

$$\bar{q}^2\bar{p}^2 + \bar{\Gamma}(\bar{q}^2 + \bar{p}^2) + 2\bar{\epsilon}\,\bar{q}\,\bar{p} \pm 1 = 0, \text{ for } \mathcal{K} \gtrless 0$$

| | a.1, a'.1 | a.3, a'.3 | a.3, a'.3 | b.1, b'.1 |
|---|---|---|---|---|
| $\mathcal{K} \in$ | $(-\infty; 0] \leq 0$ | $(-\infty; 0] \leq 0$ | $\left[0; (\lvert\epsilon\rvert - \Gamma)^2\right] \geq 0$ | $\left[0; (\lvert\epsilon\rvert + \Gamma)^2\right] \geq 0$ |
| $B$ | $\frac{k}{k'} - \frac{k'}{k} \in \mathbb{R}$ | $\frac{k}{k'} - \frac{k'}{k} \in \mathbb{R}$ | $\frac{1}{k'} + k' \geq 2$ | $\frac{1}{k} + k \geq 2$ |
| $k(B)$ | $\frac{1}{\sqrt{2}}\sqrt{1 + \frac{B}{\sqrt{B^2+4}}}$ | $\frac{1}{\sqrt{2}}\sqrt{1 + \frac{B}{\sqrt{B^2+4}}}$ | $\frac{\sqrt[4]{B^2-4}\sqrt{B - \sqrt{B^2-4}}}{\sqrt{2}}$ | $\frac{B - \sqrt{B^2-4}}{2}$ |
| $\eta$ | $\operatorname{arcds}\sqrt{\frac{k\,k'\,\Gamma}{\sqrt{\lvert\mathcal{K}\rvert}}}$ | $\operatorname{arcds}\sqrt{\frac{k\,k'\,\Gamma}{\sqrt{\lvert\mathcal{K}\rvert}}}$ | $\operatorname{arccs}\sqrt{\frac{k'\,\Gamma}{\sqrt{\mathcal{K}}}}$ | $\operatorname{arcns}\sqrt{\frac{-k\,\Gamma}{\sqrt{\mathcal{K}}}}$ |
| $u_0$ | $\operatorname{arccn}\frac{q_0}{\sqrt[4]{\lvert\mathcal{K}\rvert}}\sqrt{\frac{k'}{k}}$ | $\operatorname{arccn}\frac{q_0}{\sqrt[4]{\lvert\mathcal{K}\rvert}}\sqrt{\frac{k'}{k}}$ | $\operatorname{arcdn}\frac{q_0\sqrt{k'}}{\sqrt[4]{\mathcal{K}}}$ | $\operatorname{arcsn}\frac{q_0}{\sqrt{k}\sqrt[4]{\mathcal{K}}}$ |
| $q/\sqrt[4]{\lvert\mathcal{K}\rvert}$ | $\sqrt{\frac{k}{k'}}\operatorname{cn}[u_0 - n\,\eta]$ | $\sqrt{\frac{k}{k'}}\operatorname{cn}[u_0 - n\,\eta]$ | $\sqrt{\frac{1}{k'}}\operatorname{dn}[u_0 - n\,\eta]$ | $\sqrt{k}\operatorname{sn}[u_0 - n\,\eta]$ |
| $p/\sqrt[4]{\lvert\mathcal{K}\rvert}$ | $\sqrt{\frac{k}{k'}}\operatorname{cn}[u_0 - (n+1)\eta]$ | $\sqrt{\frac{k}{k'}}\operatorname{cn}[u_0 - (n+1)\eta]$ | $\sqrt{\frac{1}{k'}}\operatorname{dn}[u_0 - (n+1)\eta]$ | $\sqrt{k}\operatorname{sn}[u_0 - (n+1)\eta]$ |

# Plot of Jacobi elliptic functions

| | k=0 | k=0.5 | k=0.95 | k=1 |
|---|---|---|---|---|
| sn(t,k) |  |  |  |  |
| cn(t,k) |  |  |  |  |
| dn(t,k) |  |  |  |  |

# Hamiltonian for individual curve

$$\begin{aligned}
\mathrm{sn}(x \pm y) &= \frac{\mathrm{sn}x\,\mathrm{cn}y\,\mathrm{dn}y \pm \mathrm{sn}y\,\mathrm{cn}x\,\mathrm{dn}x}{1 - k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y} & \frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{sn}\,z = \mathrm{cn}\,z\,\mathrm{dn}\,z \\
\mathrm{cn}(x \pm y) &= \frac{\mathrm{cn}x\,\mathrm{cn}y \mp \mathrm{sn}x\,\mathrm{sn}y\,\mathrm{dn}x\,\mathrm{dn}y}{1 - k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y} & \frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{cn}\,z = -\mathrm{sn}\,z\,\mathrm{dn}\,z \\
\mathrm{dn}(x \pm y) &= \frac{\mathrm{dn}x\,\mathrm{dn}y \mp k^2\mathrm{sn}x\,\mathrm{sn}y\,\mathrm{cn}x\,\mathrm{cn}y}{1 - k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y} & \frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{dn}\,z = -k^2\mathrm{sn}\,z\,\mathrm{cn}\,z
\end{aligned}$$

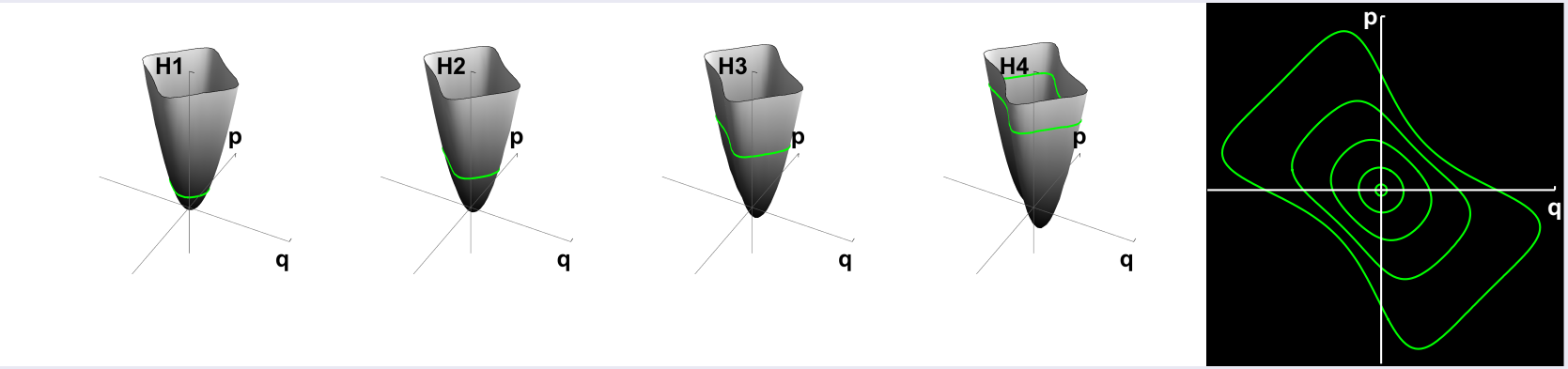
$$\boxed{H = \eta\sqrt{|\Gamma|}\left[\frac{p^2}{2} + \frac{\epsilon}{\Gamma}\,q\,p + \frac{q^2}{2} + \frac{1}{\Gamma}\frac{q^2 p^2}{2}\right] \times \begin{cases} \sqrt{k} \\ \sqrt{k\,k'} \\ \sqrt{k'} \end{cases}}$$

# 4. Action-angle coordinates

**Iatrou** and **Roberts** suggest to use $u$ and $k$ as action-angle coordinates on each canonical symmetric biquadratic curve

$$\begin{aligned} k_{n+1} &= k_n \\ u_{n+1} &= u_n + \eta \end{aligned} \quad \rightarrow \quad \begin{aligned} k_n &= k \\ u_n &= u_0 + \eta\, n \end{aligned}$$

We will look for mechanical analogies of action and angle



# 4.1 Action

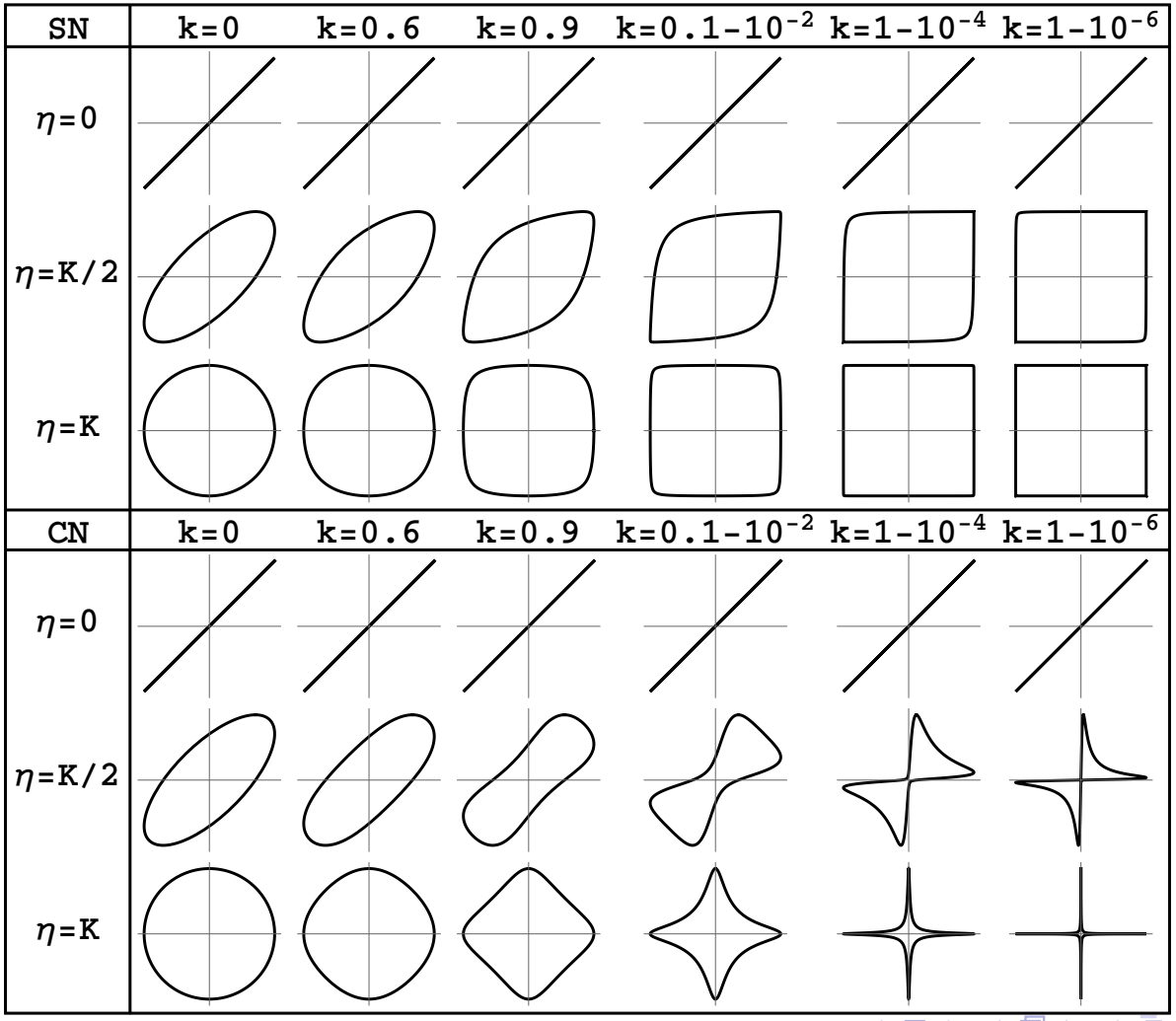
$$J = \frac{1}{2\pi} \oint p \, \mathrm{d}q = \frac{\sqrt{|\mathcal{K}|}}{2\pi} \oint \tilde{p} \, \mathrm{d}\tilde{q} = \frac{\sqrt{|\mathcal{K}|}}{2\pi} \times \begin{cases} k \;\; S_{\mathrm{sn}} \\ \frac{k}{k'} \, S_{\mathrm{cn}} \\ \frac{1}{k'} \, S_{\mathrm{dn}} \end{cases}$$

where $S_{\mathrm{sn}}$, $S_{\mathrm{cn}}$, $S_{\mathrm{dn}}$ are areas of **elliptic Lissajous curves** with commensurate frequencies $m/n = 1$

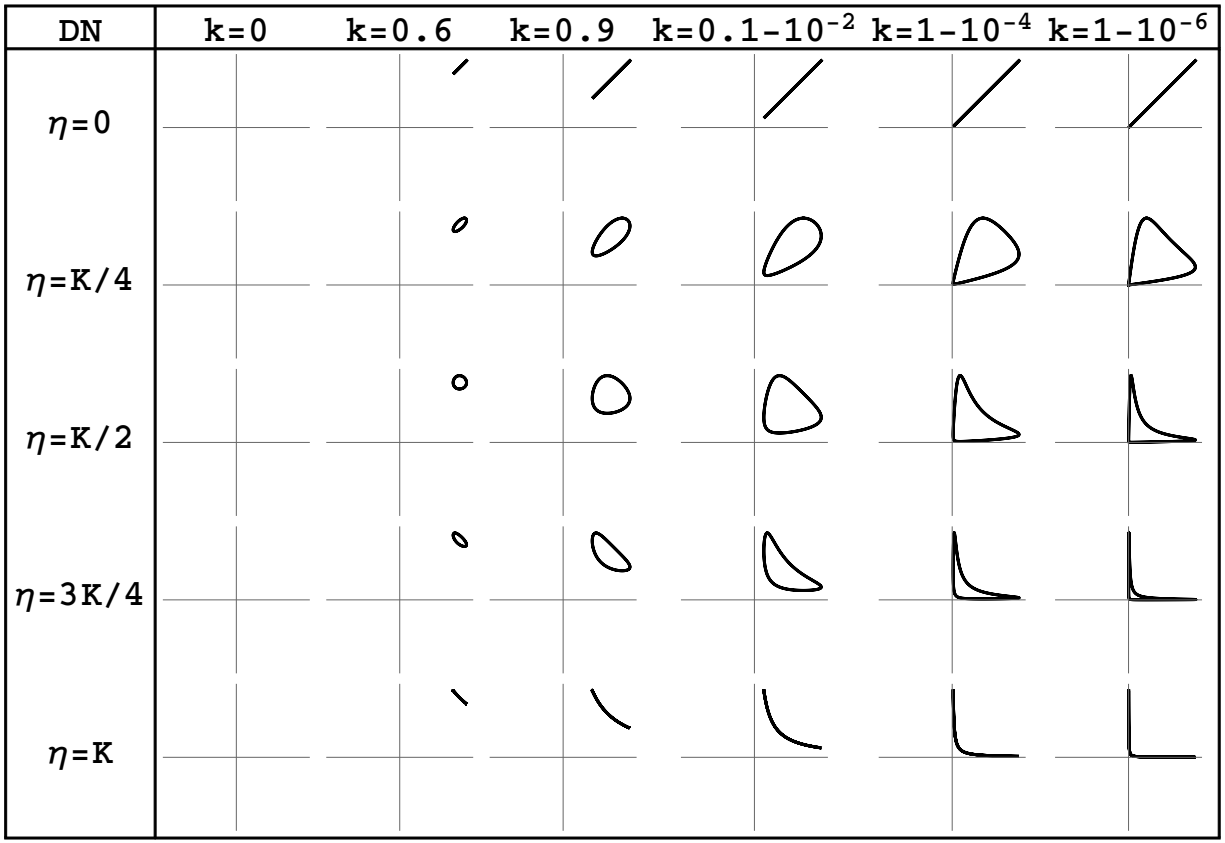
| | | |
|---|---|---|
| $\mathrm{sn}(m\,t)$ | $\mathrm{cn}(m\,t)$ | $\mathrm{dn}(m\,t)$ |
| $\mathrm{sn}(n\,t + \eta)$ | $\mathrm{cn}(n\,t + \eta)$ | $\mathrm{dn}(n\,t + \eta)$ |

$$S_{\mathrm{ef}} = \oint \mathrm{ef}(t + \eta; k) \, \mathrm{d}\,\mathrm{ef}(t; k)$$

# Elliptic Lissajous curves for SN and CN



# Elliptic Lissajous curves for DN



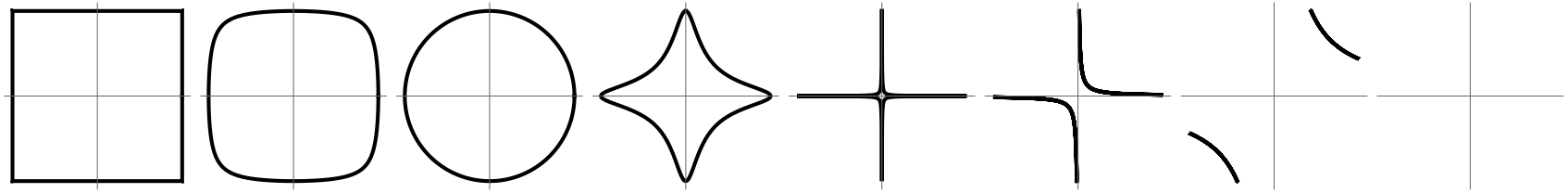
# $S_{\mathrm{sn}}$, $S_{\mathrm{cn}}$, $S_{\mathrm{dn}}$

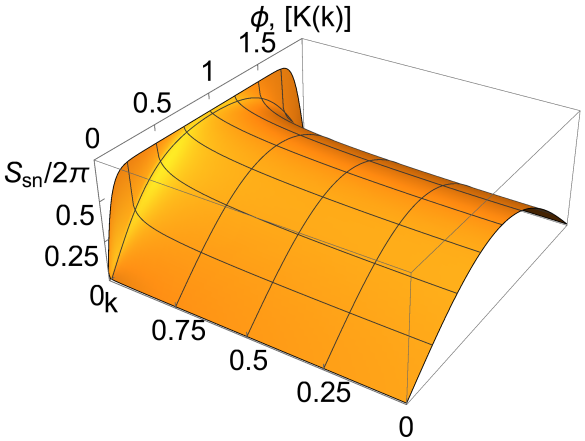
$$S_{\mathrm{sn}} = \boxed{\frac{4}{k^2}\frac{1}{\mathrm{sn}^3 y}\Big[-\mathrm{dn}^2 y\,\mathrm{K} + \mathrm{sn}^2 y\,\mathrm{E} + \mathrm{cn}^2 y\,\mathrm{dn}^2 y\,\Pi(k^2\mathrm{sn}^2 y, k)\Big]}$$

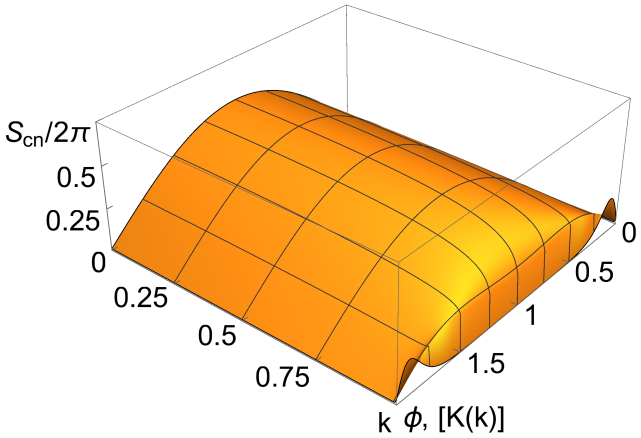
$$S_{\mathrm{cn}} = \boxed{\frac{4}{k^2}\frac{\mathrm{dn} y}{\mathrm{sn}^3 y}\Big[\mathrm{K} - \mathrm{sn}^2 y\,\mathrm{E} - \mathrm{cn}^2 y\,\Pi(k^2\mathrm{sn}^2 y, k)\Big]}$$

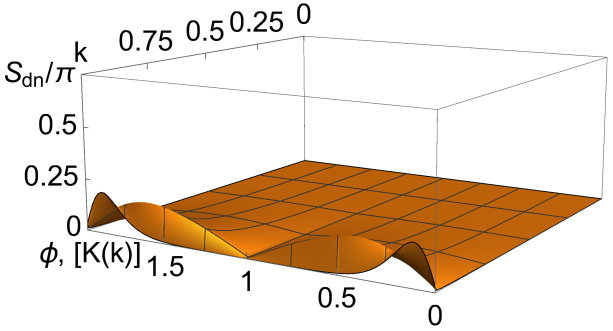
$$S_{\mathrm{dn}} = \boxed{2\,\frac{\mathrm{cn} y}{\mathrm{sn}^3 y}\Big[(\mathrm{sn}^2 y + \mathrm{dn}^2 y)\mathrm{K} - \mathrm{sn}^2 y\,\mathrm{E} - \mathrm{dn}^2 y\,\Pi(k^2\mathrm{sn}^2 y, k)\Big]}$$

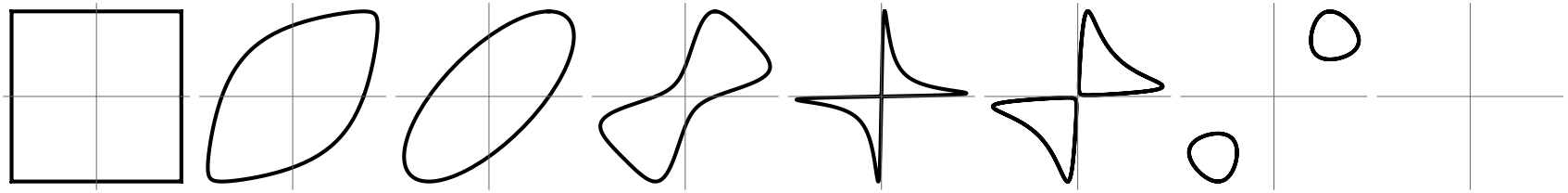
# $S_{\mathrm{sn}}$, $S_{\mathrm{cn}}$, $S_{\mathrm{dn}}$





# 5.2 Angle variable

## Brutal force — Fourier series

$$q(t) = \sqrt[4]{|\mathcal{K}|}\, f(k, k')\, \mathrm{ef}\,(u_0 + \eta\, t, k) \qquad q_n = q(n)$$

$$\mathrm{sn}\,\eta = \frac{\pi}{k\,\mathrm{K}} \sum_{n=0}^{\infty} \mathrm{csch}\left[(2n+1)\,\frac{\pi\,\mathrm{K}'}{2\,\mathrm{K}}\right] \sin\left[2\pi(2n+1)\,\boxed{\frac{\eta}{4\,\mathrm{K}}}\right]$$

$$\mathrm{cn}\,\eta = \frac{\pi}{k\,\mathrm{K}} \sum_{n=0}^{\infty} \mathrm{sech}\left[(2n+1)\,\frac{\pi\,\mathrm{K}'}{2\,\mathrm{K}}\right] \cos\left[2\pi(2n+1)\,\boxed{\frac{\eta}{4\,\mathrm{K}}}\right]$$

$$\mathrm{dn}\,\eta = \frac{\pi}{2\mathrm{K}} + \frac{\pi}{\mathrm{K}} \sum_{n=1}^{\infty} \mathrm{sech}\left[\frac{n\,\pi\,\mathrm{K}'}{\mathrm{K}}\right] \cos\left[2\pi\, n\,\boxed{\frac{\eta}{2\,\mathrm{K}}}\right]$$

**Theorem** (Danilov). Let $\mathcal{D} : \mathbb{R}^2 \to \mathbb{R}^2$ be the area-preserving integrable map which has an invariant of motion $\mathcal{K}(q,p) = \mathcal{K}(q',p')$ where $(q',p')$ is an image of $(q,p)$, i.e. $\mathcal{D}$ maps $(q,p) \mapsto (q',p')$. Then a frequency for every trajectory can be calcculated as:

$$\nu = \frac{\int_q^{q'} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}\, q}{\oint \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}\, q}.$$

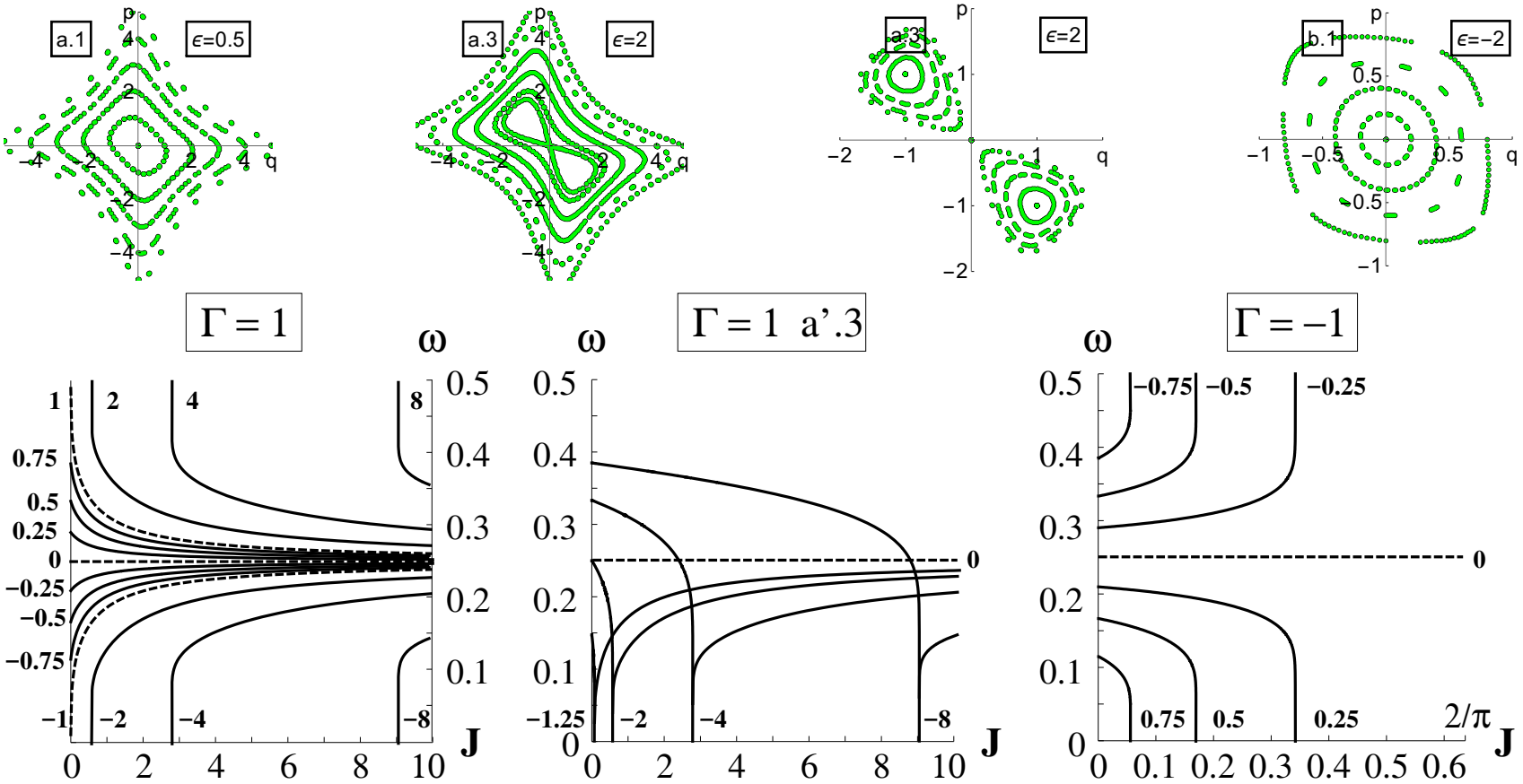
**Proof:** □

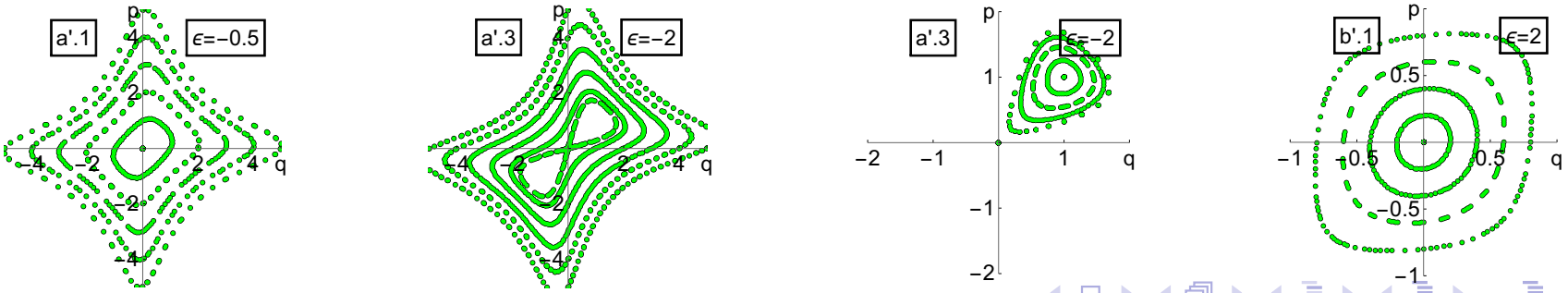
Consider assosiated continuous system with Hamiltonian $H[q,p;t]$

$$\nu = \frac{1}{T} = \frac{\int_q^{q'} \frac{\mathrm{d}t}{\mathrm{d}q}\, \mathrm{d}q}{\oint \frac{\mathrm{d}t}{\mathrm{d}q}\, \mathrm{d}q} = \frac{\int_q^{q'} \left(\frac{\partial H}{\partial p}\right)^{-1} \mathrm{d}q}{\oint \left(\frac{\partial H}{\partial p}\right)^{-1} \mathrm{d}q} = \frac{\int_q^{q'} \cancel{\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}H}} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q}{\oint \cancel{\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}H}} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q}$$

Q.E.D. ■

## Action



# 5.1 Summary

• Application of most general 1D asymmetric McMillan map for accelerator physics
• Solutions for all finite trajectories of canonical McMillan map via Jacobi elliptic functions
• Normal forms of canonical McMillan map including analytical expressions for mechanical analogies of action-angle variables
• Description of horizontal and vertical planes of 2D magnetostatic McMillan lens and any plane with zero total angular momentum for axially symmetric electron McMillan lens
• Proof and check of *Danilov Theorem*

# LAST SLIDE

**Thank you for your attention!**

**Questions?**