Wakefields and Impedance

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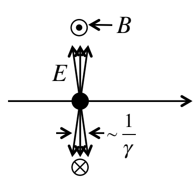
Consider the effect that one particle can have on subsequent particles through the interaction with the environment (beam pipe, RF cavities, etc)

The fields of a single particle moving relativistically are



$$E \approx \frac{q}{2\pi\epsilon_0 r}\delta(z-ct)$$
$$B \approx \frac{q}{2\pi\epsilon_0 c r}\delta(z-ct)$$

If the particle is propagating through a beam pipe, we can express the charge and current densities as (homework)

$$\sigma \approx \frac{q}{2\pi r}\delta(z-ct)\delta(r)$$
$$B \approx \frac{qc}{2\pi r}\delta(z-ct)\delta(r)$$

By symmetry, we expect only

$$E_z, E_r, \text{ and } B_\theta$$

components. We also expect the solution to propagate along the beam pipe with the particle, so we transform to

$$B_\theta(r,z,t) = \int_{-\infty}^{\infty} e^{ik(z-ct)} \tilde{B}_\theta(r) dk$$

$$E_{r,z}(r,z,t) = \int_{-\infty}^{\infty} e^{ik(z-ct)} \tilde{E}_{r,z}(r) dk$$

Recalling the appropriate Maxwell's equations, we have

$$\frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

$$-\frac{\partial B_\theta}{\partial z} = \frac{1}{c^2}\frac{\partial E_r}{\partial t}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t}$$

For any component of the field *f*, our transformation imply

$$\frac{\partial}{\partial t} f(r,z,t) = (-ikc) f(r,z,t)$$

$$\frac{\partial}{\partial z} f(r,z,t) = (ik) f(r,z,t)$$

$$\frac{\partial}{\partial r} f(r,z,t) = \int_{-\infty}^{\infty} e^{ik(z-ct)} (r) dk$$

Move the integral completely outside, and this becomes

$$\frac{1}{r}\frac{\partial(r\tilde{E}_r)}{\partial r} + ik\tilde{E}_z = \frac{\rho}{\epsilon_0} = \frac{q}{2\pi r\epsilon_0}\delta(r)$$

$$\longrightarrow \quad \frac{\partial \tilde{E}_r}{\partial r} + \frac{1}{r}\tilde{E}_r + ik\tilde{E}_z = \frac{q}{2\pi r\epsilon_0}\delta(r)$$

$$-ik\tilde{B}_\theta = -\frac{1}{c^2}(ikc)\tilde{E}_r$$

$$\longrightarrow \quad \tilde{B}_\theta = \frac{1}{c}\tilde{E}_r$$

$$ik\tilde{E}_r - \frac{\partial \tilde{E}_z}{\partial r} = (ikc)\tilde{B}_\theta$$

Combining the second and the third gives

$$\frac{\partial \tilde{E}_z}{\partial r} = 0$$
$$\tilde{E}_z = A \quad \text{(constant)}$$

Plug this into the first equation, we have

$$\frac{1}{r}\frac{\partial\left(r\tilde{E}_r\right)}{\partial r} + ikA = \frac{q}{2\pi r\epsilon_0}\delta(r)$$

Multiply through by r and integrate

$$r\tilde{E}_r + \frac{1}{2}ikAr^2 = \frac{q}{2\pi\epsilon_0}$$
$$\longrightarrow \tilde{E}_r = \frac{q}{2\pi\epsilon_0 r} - \frac{1}{2}ikAr$$

To solve for A, we look at the wall boundary, where

$$\vec{j} = \sigma\vec{E}$$

Now our equations become

$$\frac{1}{r}\frac{\partial\left(r\tilde{E}_r\right)}{\partial r} + ik\tilde{E}_z = 0$$
$$ik\tilde{E}_r - \frac{\partial \tilde{E}_z}{\partial r} = (ikc)\tilde{B}_\theta$$
$$-ik\tilde{B}_\theta = -\frac{1}{c^2}(ikc)\tilde{E}_r + \mu_0 j_r$$
$$\tilde{B}_\theta = \frac{1}{c}\left(1 + i\frac{\mu_0 c\sigma}{k}\right)\tilde{E}_r$$

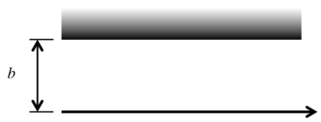
$$ik\tilde{E}_r - \frac{\partial \tilde{E}_z}{\partial r} = \frac{1}{c}\left(1 + i\frac{\mu_0 c\sigma}{k}\right)\tilde{E}_r$$
$$\longrightarrow \tilde{E}_r = \frac{1}{\mu_0 c\sigma}\frac{\partial \tilde{E}_z}{\partial r}$$

Plug this back into the first equation

$$\boxed{\mu_0 c = \frac{1}{\epsilon_0 c}} \longrightarrow \frac{1}{\mu_0 c\sigma}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{E}_z}{\partial r}\right) + ik\tilde{E}_z = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{E}_z}{\partial r}\right) + i\frac{k\sigma}{\epsilon_0 c}\tilde{E}_z = 0$$

Evaluate at the wall of the beam pipe of radius *b*



$r < b: \quad \tilde{E}_z = A \text{ (constant)}$

$r \geq b: \quad \tilde{E}_z = Ae^{-i\lambda(r-b)} \longleftarrow$ assume

Assume im{λ}>>1/r, so $\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial \tilde{E}_z}{\partial r} \approx \frac{\partial^2}{\partial r^2}\tilde{E}_z$

For r>b $\quad \tilde{E}_z = Ae^{i\lambda(r-b)} \rightarrow \frac{\partial^2}{\partial r^2}\tilde{E}_z = -\lambda^2\tilde{E}_z \rightarrow \lambda^2 = \frac{ik\sigma}{\epsilon_0 c}$

To keep solution finite, $\mathrm{Im}(\lambda) > 0$

try $\longrightarrow \lambda = \sqrt{\frac{|k|\sigma}{\epsilon_0 c}}\left(\frac{i + \mathrm{sign}(k)}{\sqrt{2}}\right)$ $\qquad$ Skin depth $\delta = \frac{1}{\mathrm{Im}(\lambda)}$

To find B_θ, use

$$\tilde{B}_\theta = \frac{1}{c}\left(1 + \frac{i\sigma}{\epsilon_0 ck}\right)\tilde{E}_z$$

$$\longrightarrow \frac{\partial \tilde{B}_\theta}{\partial r} = \frac{1}{c}\left(1 + \frac{i\sigma}{\epsilon_0 ck}\right)\frac{\partial \tilde{E}_z}{\partial r}$$

$$= \frac{1}{c}\left(1 + \frac{i\sigma}{\epsilon_0 ck}\right)(-ik)Ae^{i\lambda(r-b)}$$

Integrate and rearrange some terms

$$\tilde{B}_\theta = -\frac{1}{c}\left(\frac{\lambda}{k} + \frac{k}{\lambda}\right)Ae^{i\lambda(r-b)}$$

Matching the solutions at *r=b*, we get

$$\tilde{B}_\theta\Big|_{r=b} = -\frac{1}{c}\left(\frac{\lambda}{k} + \frac{k}{\lambda}\right)A$$

$$= \frac{1}{c}\left(1 + \frac{i\sigma}{c\epsilon_0 k}\right)\tilde{E}_r\Big|_{r=b}$$

$$= \frac{1}{c}\left(1 + \frac{i\sigma}{c\epsilon_0 k}\right)\left(\frac{q}{2\pi\epsilon_0} - \frac{1}{2}ikAb\right)$$

Solve for *A*, and after a bit of algebra, we get

$$A = \frac{q}{2\pi\epsilon_0 b\left[\frac{1}{2}ikb - \frac{k}{\lambda} - \frac{\lambda}{k}\right]}$$

$\sim \frac{1}{k\delta} \rightarrow$ very large

$$\approx -\frac{qk}{2\pi\epsilon_0 b\lambda}$$

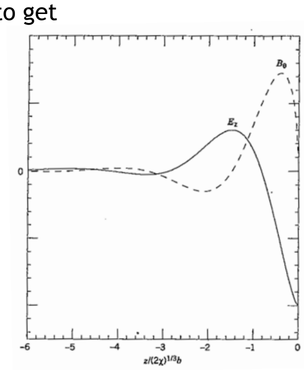
We now Fourier transform back into the lab frame to get

$$E_z = \sqrt{\frac{c}{2\pi\epsilon_0}}\frac{q}{b}\frac{1}{|z-ct|^{3/2}}\left[1-\mathrm{H}(z-ct)\right]$$

=0 ahead of particle

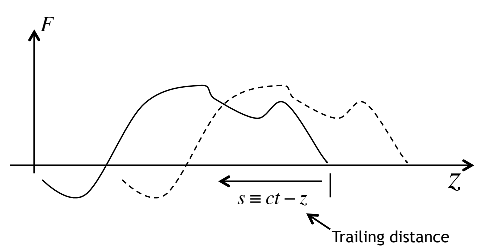
$$E_r = -\frac{3}{4}\sqrt{\frac{c}{2\pi\epsilon_0}}\frac{q}{b}\frac{r}{|z-ct|^{5/2}}\left[1-\mathrm{H}(z-ct)\right]$$

$$B_\theta = \frac{1}{c}E_r$$



Wake Functions

We are looking for a function which describes the effect that particles have on the subsequent particles. We will look for the average fields created by particles in their wake.



This will represent the average forces that a particle trailing the lead particle will experience a distance s behind the lead particle, due the wakefields it creates. The total time derivative is thus zero

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial z}$$

So, for example, the r component of $\vec{\nabla}\times\vec{B} = -\frac{\partial \vec{B}}{\partial t}$ is

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t}$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} = \frac{\partial}{\partial z}\left(E_\theta + cB_r\right)$$

convert to z derivative

In terms of forces

$$\begin{aligned} F_z &= eE_z \\ F_\theta &= eE_\theta + ecB_r \\ &= 0 + ecB_r \end{aligned}$$

$$\longrightarrow \quad \frac{1}{r}\frac{\partial F_z}{\partial \theta} = \frac{\partial F_\theta}{\partial z} F_\theta = -\frac{\partial}{\partial s} F_\theta$$

We can likewise show (homework) that

$$ec\frac{\partial B_z}{\partial r} = \frac{\partial F_\theta}{\partial z} = \frac{1}{r}\frac{\partial F_z}{\partial \theta} = -\frac{\partial F_\theta}{\partial s}$$

$$-\frac{ec}{r}\frac{\partial B_z}{\partial r} = \frac{\partial F_r}{\partial z} = \frac{\partial F_z}{\partial r} = -\frac{\partial F_r}{\partial s}$$

We can write a general solution as

$$\begin{aligned} F_r &= eQ_m m r^{m-1}\cos(m\theta)W_m(s) \\ F_\theta &= -eQ_m m r^{m-1}\sin(m\theta)W_m(s) \\ F_z &= -eQ_m r^m \cos(m\theta)W_m'(s) \\ ecB_z &= Q_m r^m \sin(m\theta)W_m'(s) \end{aligned}$$

Verify for the r direction. We want

$$\frac{1}{r}\frac{\partial F_z}{\partial \theta} = -\frac{\partial}{\partial s}F_\theta$$

Check

$$\frac{1}{r}\frac{\partial F_z}{\partial \theta} = eQ_m r^{m-1}\sin(m\theta)W_m'(s)$$

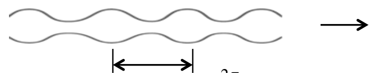
$$-\frac{\partial}{\partial s}F_\theta = eQ_m r^{m-1}\sin(m\theta)W_m'(s) \quad ✔$$

W(s) and W'(s) are called the wake functions. Often, W(s) is referred to as the "transverse wake function" and W'(s) as the "longitudinal wake function".

$m =$ "mode number"

$Q_m =$ charge contributing to that mode (watch units!)

Assume a harmonic component of the beam structure



$$\lambda \rightarrow k = \frac{2\pi}{\lambda}$$

Propagating form
$$I = I_0 e^{ik(z-ct)} = I_0 e^{i(kz-\omega t)}$$

$\omega = \frac{c}{k}$

= angular frequency at fixed point

Consider the 0 mode in the longitudinal direction

$$F_z = -qQ_0 W_0'(s) = eE_z$$

So we can write the field induced by the charges in front of it as

$E_z(z,t)$

$$
\begin{aligned}
E_z(z,t) &= -\int dq\left(z,t-\frac{s'}{c}\right)W_0'(s)\\
&= -\int \frac{dq}{dt}dt W_0'(s)\\
&= -\int I\frac{ds}{c}W_0'(s)\\
&= -\int_0^\infty I\left(z,t-\frac{s'}{c}\right)W_0'(s)\frac{ds'}{c}\\
&= -\int_0^\infty I_0 e^{i\left[kz-\omega\left(t-\frac{s'}{c}\right)\right]}W_0'(s)\frac{ds'}{c} = -\int_0^\infty I_0 e^{i(kz-\omega t)}e^{i\frac{\omega s'}{c}}W_0'(s)\frac{ds'}{c}\\
&= -I_0(z,t)\int_0^\infty e^{i\frac{\omega s'}{c}}W_0'(s)\frac{ds'}{c}
\end{aligned}
$$

appears to be a Fourier Transform

Recalling our discussion of the negative mass instability, we define an impedance

$$V = E_z L \equiv -I(z,t) Z^{\parallel}$$

We can identify

$$\frac{Z_0^{\parallel}}{L} = \frac{1}{c}\int_{-\infty}^{\infty} e^{i\omega\frac{s'}{c}} W_0'(s')ds'$$

$$V(z,t) = -I_0(x,t) Z_0^{\parallel}$$

Generalize

$$\frac{Z_m^{\parallel}}{L} = \frac{1}{c}\int_{-\infty}^{\infty} e^{i\omega\frac{s'}{c}} W_m'(s')ds' \qquad \left(\text{units}=\frac{[\Omega]}{[L]^m}\right)$$

$$\frac{Z_m^{\perp}}{L} = \frac{1}{ic}\int_{-\infty}^{\infty} e^{i\omega\frac{s'}{c}} W_m(s')ds'$$

convention, because transverse fields tend to be out of phase

Calculating wakefields and impedances can be very difficult, even in simple geometries; however, we'll see that if we know they exist, we can say something about their effects, and also about how to measure them.