COLLECTIVE EFFECTS

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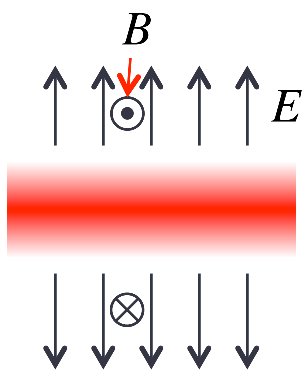
Space Charge

So far, we have not considered the effect that particles in a bunch might have on each other, or on particles in another bunch.

Consider the effect off space charge on the transverse distribution of the beam.



$$\rho(r) = \frac{Ne}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

radial charge density

If we look at the field at a radius r, we have

$$\oint \vec{E} \bullet d\vec{A} = E(2\pi r L) = \frac{Q_{encl}}{\epsilon_0} = \frac{Ne}{\sigma^2} \int_0^r r e^{-r^2/2\sigma^2} \, dr$$

$$= Ne\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$\longrightarrow \vec{E} = \frac{Ne}{2\pi\epsilon_0 r L}\left(1 - e^{-r^2/2\sigma^2}\right)\hat{r}$$

Similarly, Ampere's Law gives

$$\oint \vec{B} \bullet d\vec{l} = 2\pi r B = \mu_0 I_{enclosed} = \mu_0 \frac{Nev}{\sigma^2 L} \int_0^r r e^{-r^2/2\sigma^2}\, dr$$

$$\longrightarrow \quad \vec{B} = \mu_0 \frac{Nev}{2\pi r L}\left(1 - e^{-r^2/2\sigma^2}\right)\hat{\theta}$$

$$\longrightarrow \quad \vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$= \frac{Ne^2}{2\pi L}\left(1 - e^{-r^2/2\sigma^2}\right)\left(\frac{1}{\epsilon_0}\hat{r} + \underbrace{v^2\mu_0}_{=\frac{1}{\epsilon_0}\left(\epsilon_0\mu_0\right) = \frac{1}{\epsilon_0}\frac{1}{c^2}}\left(\underbrace{\hat{s} \times \hat{\theta}}_{=-\hat{r}}\right)\right)$$

$$= \hat{r}\frac{Ne^2}{2\pi r L \epsilon_0}\left(1 - e^{-r^2/2\sigma^2}\right)\left(1 - \beta^2\right)$$

$$= \hat{r}\frac{ne^2}{2\pi r \epsilon_0 \gamma^2}\left(1 - e^{-r^2/2\sigma^2}\right); \quad \mathrm{n} \equiv \frac{N}{L} = \frac{dN}{ds}$$

Linear charge density

We can break this into components in x and y

$$F_x = |F|\frac{x}{r}$$

$$= \frac{ne^2}{2\pi\epsilon_0\gamma^2}\frac{x}{r^2}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$= \frac{ne^2}{2\pi r\epsilon_0\gamma^2}\frac{x}{(x^2+y^2)}\left(1-e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$$

$$F_y = \frac{ne^2}{2\pi r\epsilon_0\gamma^2}\frac{y}{(x^2+y^2)}\left(1-e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$$

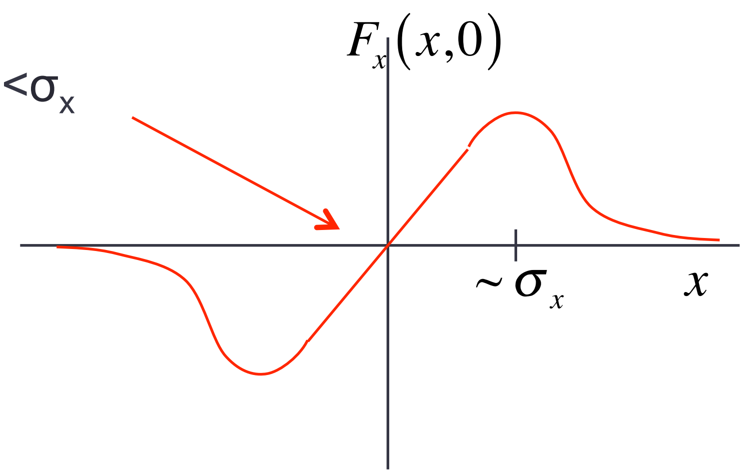
Non-linear and coupled ➔ ouch! but for $x \ll \sigma_x$

$$\left(1-e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) \approx \frac{(x^2+y^2)}{2\sigma^2}$$

$$\rightarrow \quad F_x \approx \frac{ne^2}{4\pi\sigma^2\epsilon_0\gamma^2}x$$

$$F_y \approx \frac{ne^2}{4\pi\sigma^2\epsilon_0\gamma^2}y$$

~linear and decoupled



$$F_x = \frac{dp_x}{dt}$$

$$\rightarrow \Delta x' = \frac{\Delta p_x}{p} = \frac{1}{p}\int F_x\,dt = \frac{1}{p}\int F_x \frac{dt}{ds}ds$$

$$= \frac{1}{vp}\int F_x\,ds$$

$$x'' = \frac{F_x}{vp} = \frac{F_x}{\beta^2\gamma m c^2}$$

$$\approx \frac{e^2}{4\pi\sigma^2\epsilon_0\beta^2\gamma^3}nx$$

$$= \frac{r_0}{\beta^2\gamma^3\sigma^2}nx; \quad \mathrm{r}_0 \equiv \frac{e^2}{4\pi\epsilon_0 m_0 c^2}$$

"classical radius" $= 1.53\times 10^{-18}$ m for protons

This looks like a distributed defocusing quad of strength

$$\frac{d\left(\frac{1}{f}\right)}{ds} \equiv k = -\frac{nr_0}{\beta^2\gamma^3\sigma^2}$$

so the total tuneshift is

$$\Delta\nu_x = \frac{1}{4\pi}\oint k\beta_x(s)ds$$

$$= -\frac{r_0}{4\pi\beta^2\gamma^3}\oint n\frac{\beta_x(s)}{\sigma_x^2}ds = -\frac{r_0}{4\pi\beta^2\gamma^3\epsilon}\oint n\,ds$$

$$= \frac{1}{\epsilon_x}$$

("Off-momentum particles" lecture)

$$= -\frac{r_0}{4\pi\beta^2\gamma^3}\frac{NB}{\epsilon_x}; \quad \mathrm{B} \equiv \frac{n_{peak}}{\langle n\rangle}$$

"Bunching factor"

$$= -\frac{NBr_0}{4\pi\beta\gamma^2 L}\frac{L}{(\beta\gamma\epsilon_x)}$$

$$= \epsilon_{x,N}$$

$$= -\frac{NBr_0}{4\pi\beta\gamma^2\epsilon_{x,N}}$$

Maximum tuneshift for particles near core of beam

Example: Fermilab Booster@Injection

$$K = 400 \text{ MeV}$$

$$\text{N}=5\times 10^{12}$$

$$\epsilon_N = 2\ \pi\text{-mm-mr}$$

$$\text{B=1 (unbunched beam)}$$

$$\Delta_\nu = -\frac{Nr_0}{4\pi\beta\gamma^2\epsilon_N} = -.247$$

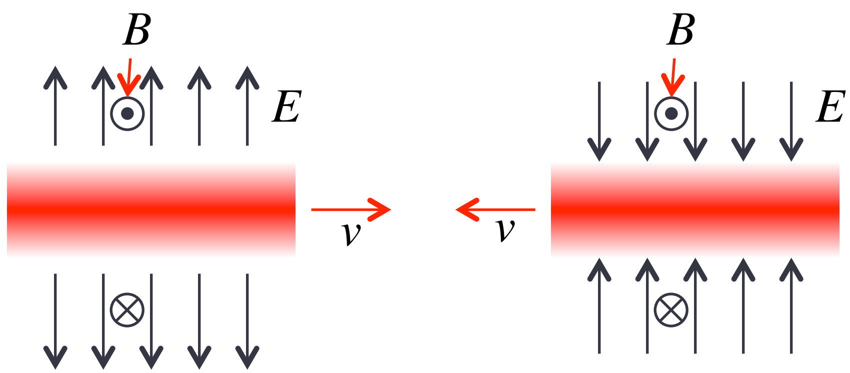
This is pretty large, but because this is a rapid cycling machine, it is less sensitive to resonances

Because this affects individual particles, it's referred to as an "incoherent tune shift", which results in a tune spread. There is also a "coherent tune shift", caused by images charges in the walls of the beam pipe and/or magnets, which affects the entire bunch more or less equally.

This is an important effect, but beyond the scope of this lecture.

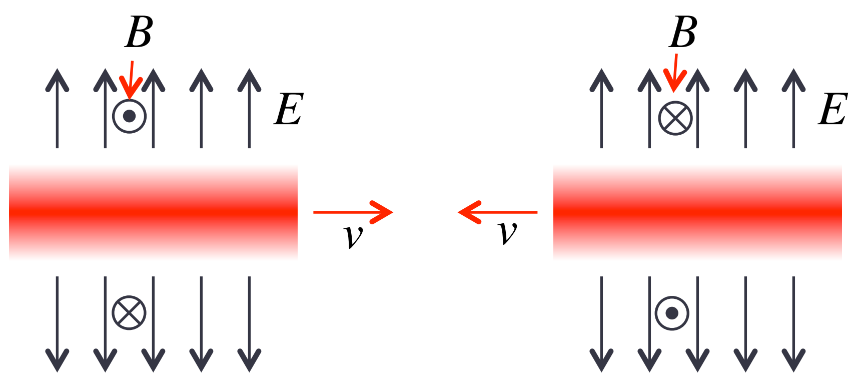
Beam-beam Interaction

If two *oppositely* charged bunches pass through each other…



Both E and B fields are *attractive* to the particles in the other bunch

If two bunches with the *same* sign pass through each other…



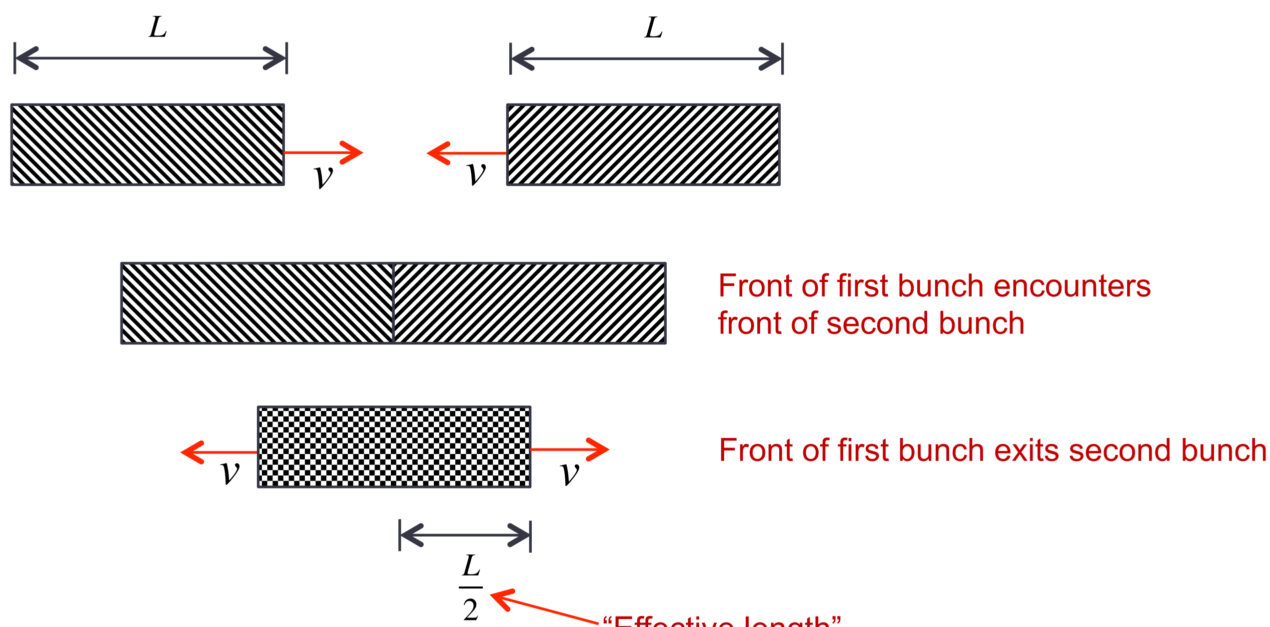
Both E and B fields are *repulsive* to the particles in the other bunch

In either case, the forces add

$$\vec{F} = -\hat{r}\frac{e^2}{2\pi\epsilon_0 r}\frac{N}{L}\left(1-e^{-r^2/2\sigma^2}\right)\overbrace{\left(1+\beta^2\right)}^{\approx 2}$$

$$\approx -\hat{r}\frac{e^2}{\pi\epsilon_0 r}\frac{N}{L}\left(1-e^{-r^2/2\sigma^2}\right)$$

Effective Length



$$\Delta x' = \frac{F_x}{vp}\Delta s = \frac{F_x}{vp}\left(\frac{L}{2}\right)$$

$$= -\frac{N_b e^2}{2\pi\epsilon_0 r\gamma\beta^2 mc^2}\frac{x}{r}\left(1-e^{-r^2/2\sigma^2}\right)$$

$\beta \approx 1$

$$\approx -\frac{2N_b r_0}{\gamma}\frac{x}{r^2}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$= -\frac{2N_b r_0}{\gamma}\frac{x}{\left(x^2+y^2\right)}\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$

$$\approx -\frac{N_b r_0}{\gamma\sigma^2}x = -\frac{1}{f_{eff}}x$$

Small x and y

$$\Delta y' \approx -\frac{N_b r_0}{\gamma\sigma^2}y = -\frac{1}{f_{eff}}y$$

$$\Delta\nu = \frac{1}{4\pi}\frac{\beta^*}{f_{eff}}$$

$$= \frac{N_b r_0}{4\pi}\frac{\beta^*}{\gamma\sigma^2} = \frac{N_b r_0}{4\pi\gamma\epsilon}$$

$$= \frac{N_b r_0}{4\pi\epsilon_N}$$

normalized emittance (protons)

$$\boxed{\equiv \xi}$$

“Tuneshift Parameter”

Maximum tuneshift for particles near center of bunch

Luminosity and Tuneshift

The total tuneshift will ultimately limit the performance of any collider, by driving the beam onto an unstable resonance. Values of on the order ~.02 are typically the limit. However, we have seen the somewhat surprising result that the tuneshift

$$\xi = \frac{N_b r_0}{2\pi\epsilon\gamma}$$

does not depend on β^*, but only on

$$\frac{N_b}{\epsilon} \equiv \quad \text{"brightness"}$$

For a collider, we have

$$\mathcal{L} = \frac{fn_b N_b^2}{4\pi\sigma^2} = \frac{fn_b N_b^2}{4\pi\left(\frac{\beta^*\epsilon_N}{\gamma}\right)} = \frac{fn_b N_b \gamma}{r_0\beta^*}\left(\frac{r_0}{4\pi}\frac{N_b}{\epsilon_N}\right)$$

$$= f\frac{n_b N_b \gamma}{r_0\beta^*}\xi$$

We assume we will run the collider at the “tuneshift limit”, in which case we can increase luminosity by

- Making β^* as small as possible
- Increasing N_b and ε proportionally.