Gravitational Bending of Light

Don Edwards

May 24, 2007

1 An Easy but Wrong Approach

Anyone who has done a small angle scattering calculation in elementary nuclear physics might try the same thing for a photon passing close by the sun. After all, in applying Newton's Law of Gravity to the transverse deflection of a photon, one isn't trying to change its speed, only its direction. The result for the angle of deflection, α, is

$$\alpha = \frac{2GM_\odot}{c^2 R_\odot} = 0.87 \text{ arcseconds}, \tag{1}$$

for $M_\odot = 2 \times 10^{30}$ kg, $R_\odot = 7 \times 10^8$ m, and $G = (20/3) \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-3} \mathrm{s}{-2}$. The solar radius plays the role of impact parameter.

Unfortunately, the answer given by Eq. **??** is off by exactly a factor of two; the actual deflection is twice as large. Interestingly enough, Einstein published the same prediction in 1908; it was not until his definitive paper on the General Theory of Relativity in 1915 that he presented the 1.75 arcsecond calculation. Einstein had a good excuse for the 1908 error, for there were not measurements with which to compare. It was not until 1919 that the first attempt was made by Eddington during a total eclipse of the sun, and the results were not decisive. But today, the same excuse isn't available to us – the deflection is known experimentally to about one part in 10^5.

It's curious that Eq. **??** is wrong by exactly a factor of two, particularly since a similar Newtonian calculation of the gravitational redshift gives the right answer. The answer is to be found in a comparison with Einstein's 1915 treatment, and that is the subject of the next section.

2 A Difficult but Correct Approach

In his 1915 paper, Einstein arrives at his celebrated field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor. In addition to appearing explicity, the metric tensor and its derivatives up to second order are buried inside the Ricci tensor, $R_{\mu\nu}$ and in its contraction R. The "driving terms" are provided by the stress-energy tensor $T_{\mu\nu}$ on the right-hand side.

Einstein treated the problem in the weak field approximation, in which the metric suffers a perturbation of the form

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{u\nu} \tag{3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric of Special Relativity and $h_{\mu\nu} \ll 1$. A static mass M (energy Mc^2) is located at the origin. He found the solution

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) d\ell^2, \tag{4}$$

where $d\ell = (dx^2 + dy^2 + dz^2)^{1/2}$. The speed of light in these coordinates is found by setting $ds = 0$:

$$v \equiv \frac{d\ell}{dt} = c \left(\frac{1 - \frac{2GM}{c^2 r}}{1 + \frac{2GM}{c^2 r}} \right)^{1/2} \approx c \left(1 - \frac{2GM}{c^2 r}\right). \tag{5}$$

Thus, the speed of light as measured in this non-inertial coordinate system becomes less the closer the approach to the mass; there is in effect an index of refraction. He used Huygen's principle to calculate the deflection. If the light is headed in the x-direction in the $z = 0$ plane, then the transverse deflection develops according to

$$\frac{d\alpha}{dx} = \frac{1}{c}\frac{dv}{dy} = \frac{2GM}{c^2}\frac{y}{\ell^3} = \frac{2GM}{c^2}\frac{y}{(x^2 + y^2)^{3/2}}, \tag{6}$$

which upon integration with respect to x yields

$$\alpha = \frac{4GM}{c^2 R} \tag{7}$$

where R is the distance of closest approach to the mass. This result is twice that given by Eq. **??**, and when applied to grazing passage of the sun gives the correct answer of 1.75 arcseconds. Each of the $h_{\mu\nu}$ is at most $\approx 4 \times 10^{-6}$, so the weak field approximation is quite reasonable.

Note that equal contributions are made by both the space and time perturbations of the metric, whereas the gravitational redshift makes use of only the perturbation of the time term. The prediction of the gravitational redshift requires only the principle of equivalence; no appeal to the field equations is needed.

3 An Even More Correct Approach

In 1916, Schwarzchild published his exact solution to the field equations for the centrally located point mass problem. The Schwarzchild line element is

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{8}$$

where the element of solid angle, $d\Omega$, is given by $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The ubiquitous factors of $2GM/c^2r$ again appear, but now there is no constraint that they be small compared with unity.

This form of the line element doesn't treat the space coordinates alike, as does Eq. **??**. In order to bring them into correspondence, apply to Eq. **??** the change of radial variable

$$r = R\left(1 + \frac{GM}{2c^2R}\right)^2. \tag{9}$$

Then Eq. **??** becomes

$$ds^2 = \left(\frac{1 - \frac{GM}{2c^2R}}{1 + \frac{GM}{2c^2R}}\right)^2 c^2dt^2 - \left(1 + \frac{GM}{2c^2R}\right)^4 \left(dR^2 + R^2d\Omega^2\right). \tag{10}$$

For $GM/2c^2R \ll 1$, this last form of the line element becomes the same as Eq. **??**.

The use of Huygen's principle by Einstein has intuitive appeal, but one can calculate the light trajectory using a null geodesic based on any of the forms of the line element. The Schwarchild solution permits R to approach $2GM/c^2$, the Schwarzchild radius and the subject of black holes, but that's a different story.